MLSP linear algebra refresher


"YOU LEARN SOMETHING NEW EVERYDAY"
FALSE.
YOU LEARN SOMETHING OLD EVERY DAY. JUST BECAUSE YOU'VE JUST LEARNED IT DOESN'T MEAN IT'S NEW, OTHER PEOPLE ALREADY KNEW IT.
quickmeme.com

I learned
something old
today!

Book

- Fundamentals of Linear Algebra, Gilbert Strang

- Important to be very comfortable with linear algebra
  - Appears repeatedly in the form of Eigen analysis, SVD, Factor analysis
  - Appears through various properties of matrices that are used in machine learning
    - Often used in the processing of data of various kinds
    - Will use sound and images as examples

- Today's lecture: Definitions
  - Very small subset of all that's used
  - Important subset, intended to help you recollect

Incentive to use linear algebra

- Simplified notation!

$$\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{y} \longleftrightarrow \sum_j y_j \sum_i x_i a_{ij}$$

- Easier intuition
  – *Really convenient geometric interpretations*

- Easy code translation!

```
for i=1:n
  for j=1:m
    c(i)=c(i)+y(j)*x(i)*a(i,j)
  end
end
```

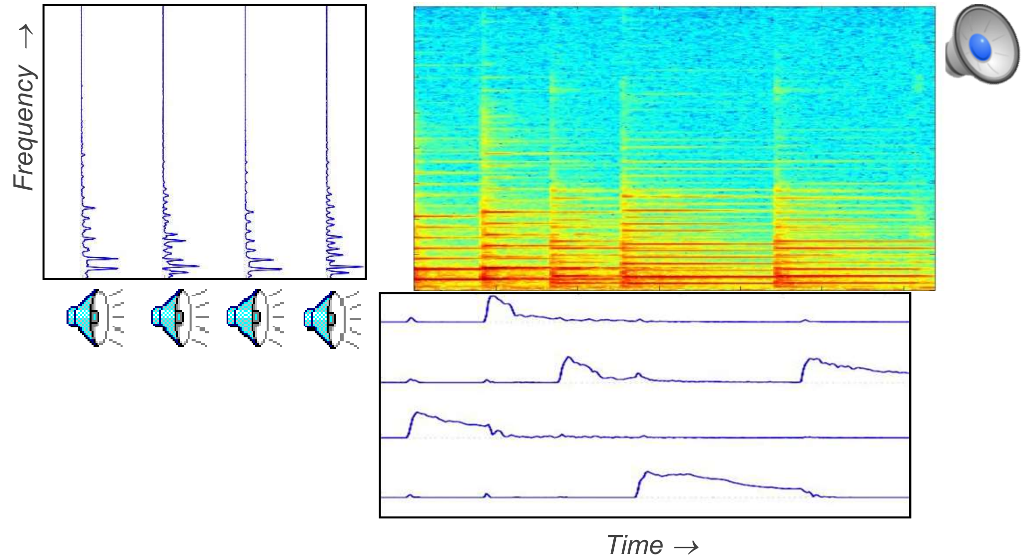
$\longleftrightarrow$

```
C=x*A*y
```

And other things you can do

Rotation + Projection +
Scaling + Perspective

From Bach's Fugue in Gm



Decomposition (NMF)

- Manipulate Data
- Extract information from data
- Represent data..
- Etc.

Overview

- Vectors and matrices
- Basic vector/matrix operations
- Various matrix types
- Matrix properties
  - Determinant
  - Inverse
  - Rank
- Solving simultaneous equations
- Projections
- Eigen decomposition
- SVD

Overview

- **Vectors and matrices**
- **Basic vector/matrix operations**
- Various matrix types
- Matrix properties
  - Determinant
  - Inverse
  - Rank
- Solving simultaneous equations
- Projections
- Eigen decomposition
- SVD

What is a vector

Column vector

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

An Nx1 vector

$$\begin{bmatrix} a & b & c \end{bmatrix}$$

Row vector

A 1xN vector

- A rectangular or horizontal arrangement of numbers

What is a vector

Column vector

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

An Nx1 vector

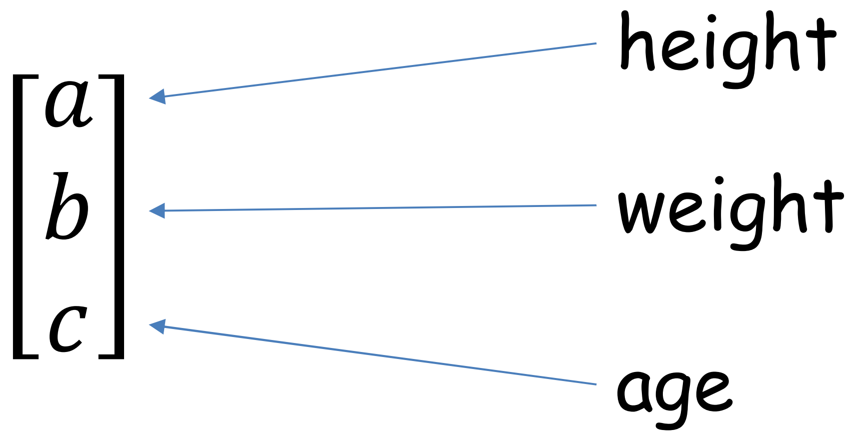
$$\begin{bmatrix} a & b & c \end{bmatrix}$$

Row vector

A 1xN vector

- A rectangular or horizontal arrangement of numbers
- Which, without additional context, is actually a useless and meaningless mathematical object

A *meaningful* vector

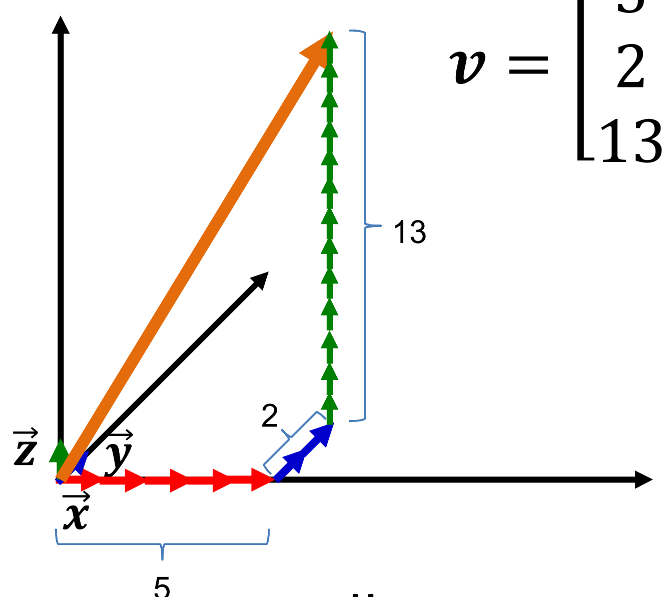


- A rectangular or horizontal arrangement of numbers
- Where each number refers to a different quantity

What is a vector

$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\boldsymbol{v} = a\vec{\boldsymbol{x}} + \mathrm{b}\vec{\boldsymbol{y}} + c\vec{\boldsymbol{z}}$$



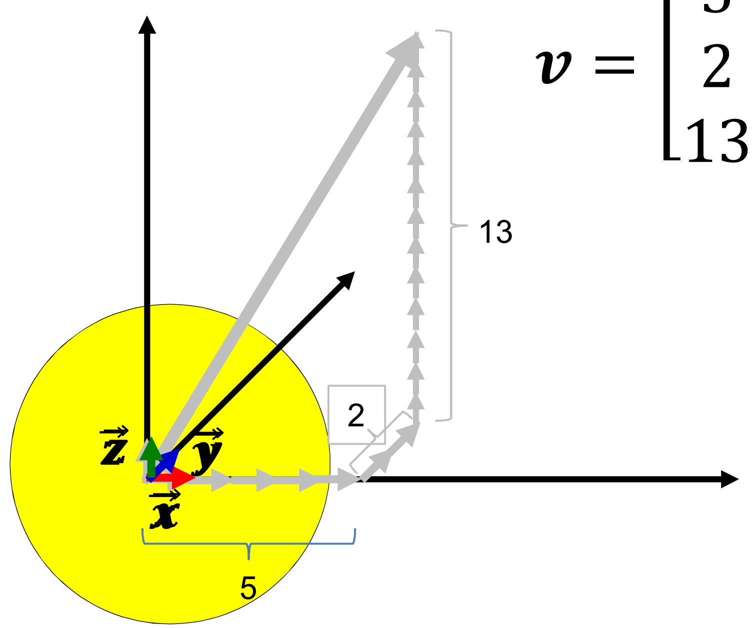
$$\boldsymbol{v} = \begin{bmatrix} 5 \\ 2 \\ 13 \end{bmatrix}$$

- Each component of the vector actually represents the *number of steps* along a set of *basis* directions
  - The vector cannot be interpreted without reference to the bases!!!!!
  - The bases are often *implicit* – we all just agree upon them and don't have to mention them

Standard Bases

$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\boldsymbol{v} = a\vec{\boldsymbol{x}} + \mathrm{b}\vec{\boldsymbol{y}} + c\vec{\boldsymbol{z}}$$



$$\boldsymbol{v} = \begin{bmatrix} 5 \\ 2 \\ 13 \end{bmatrix}$$

- "Standard" bases are "Orthonormal"
  - Each of the bases is at 90° to every other basis
    - Moving in the direction of one basis results in *no* motion along the directions of other bases
  - All bases are unit length

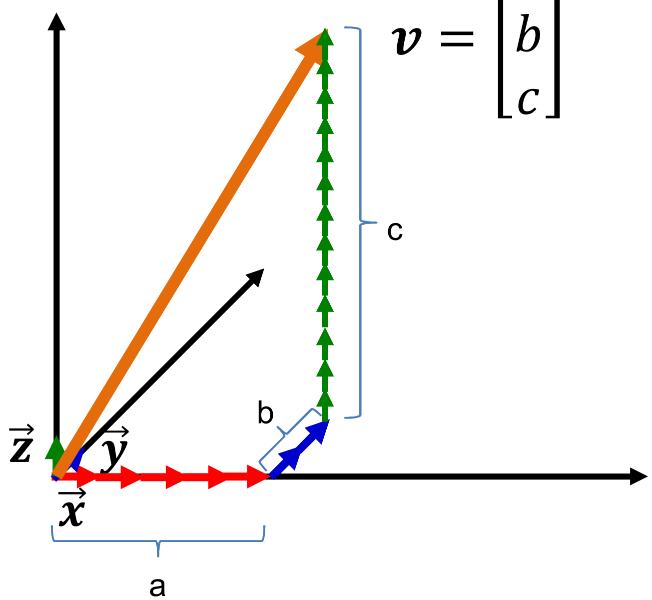
A vector by another basis..

$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{using } \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}, \vec{\boldsymbol{z}}$$

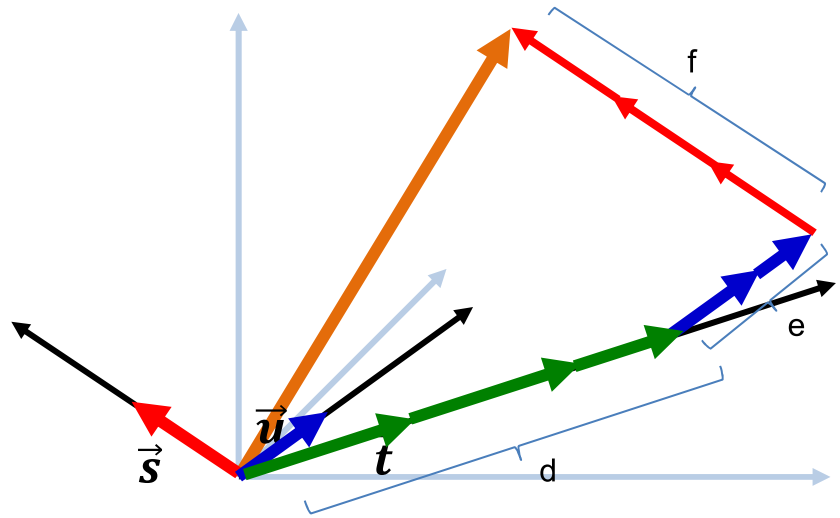
$$\boldsymbol{v} = a\vec{\boldsymbol{x}} + b\vec{\boldsymbol{y}} + c\vec{\boldsymbol{z}}$$

$$\boldsymbol{v} = d\vec{\boldsymbol{s}} + e\vec{\boldsymbol{t}} + f\vec{\boldsymbol{u}}$$

$$\boldsymbol{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$



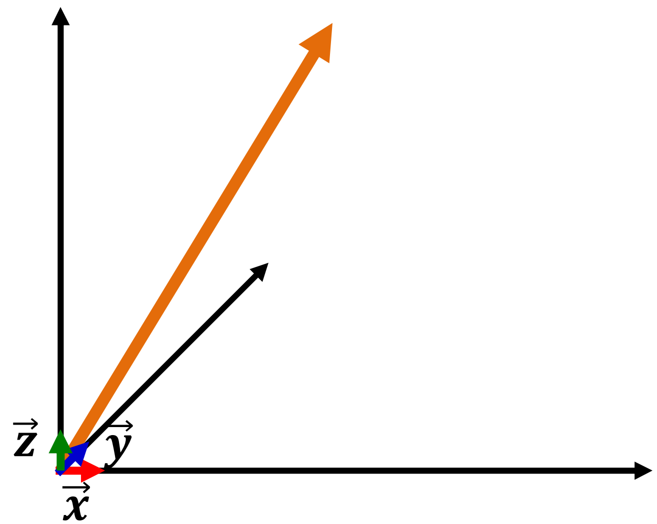
$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



- For non-standard bases we will generally *have* to specify the bases to be understood

Length of a vector

$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



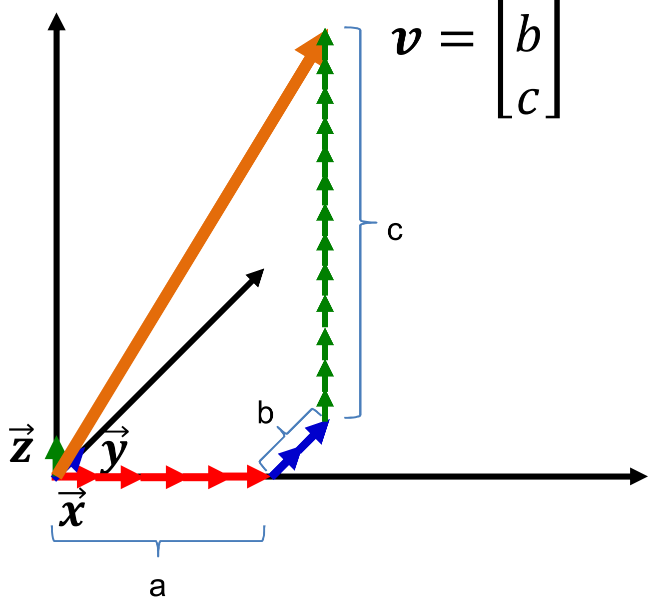
$$|\boldsymbol{v}| = \sqrt{a^2 + b^2 + c^2}$$

- The Euclidean distance from origin to the location of the vector

Length of a vector..

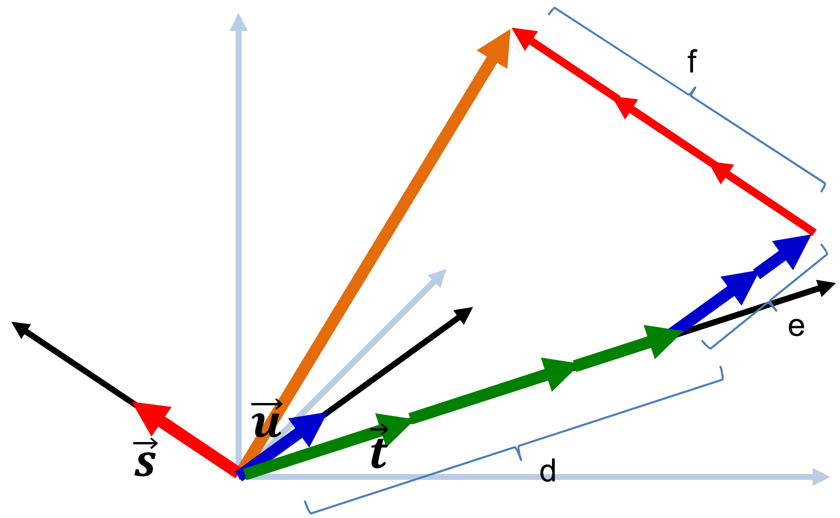
$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{using } \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}, \vec{\boldsymbol{z}}$$

$$\boldsymbol{v} = a\vec{\boldsymbol{x}} + b\vec{\boldsymbol{y}} + c\vec{\boldsymbol{z}}$$



$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\boldsymbol{v} = d\vec{\boldsymbol{s}} + e\vec{\boldsymbol{t}} + f\vec{\boldsymbol{u}} \qquad \boldsymbol{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

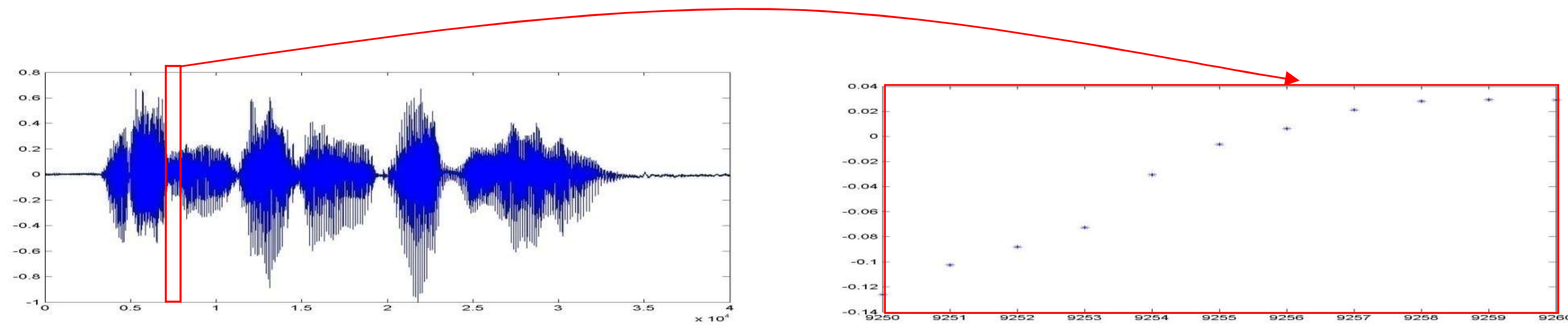


The norm of a vector depends on the bases used to specify it

$$|\boldsymbol{v}| = \sqrt{a^2 + b^2 + c^2} \quad \textbf{OR} \quad |\boldsymbol{v}| = \sqrt{d^2 + e^2 + f^2}$$

Representing signals as vectors

- Signals are frequently represented as vectors for manipulation
- E.g. A segment of an audio signal

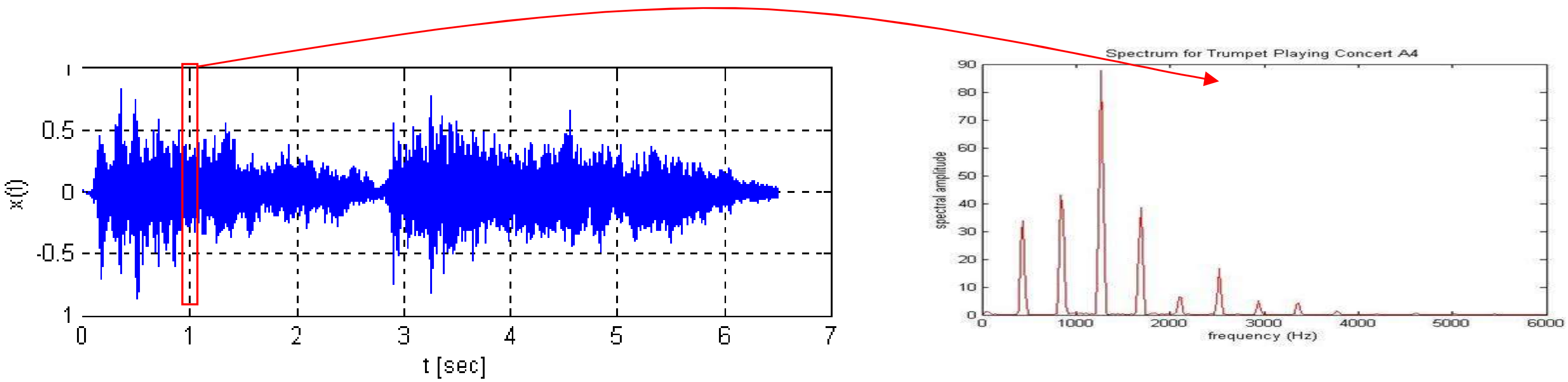


- Represented as a vector of sample values

$$[s_1 \;\; s_2 \;\; s_3 \;\; s_4 \; \ldots \; s_N]$$

Representing signals as vectors

- Signals are frequently represented as vectors for manipulation
- E.g. The ***spectrum*** segment of an audio signal

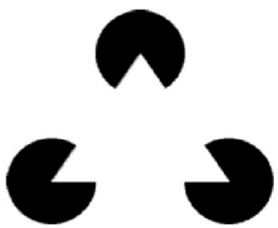


- Represented as a vector of sample values

$$[S_1 \;\; S_2 \;\; S_3 \;\; S_4 \; \ldots \; S_M]$$

  – Each component of the vector represents a frequency component of the spectrum

Representing an image as a vector



- 3 pacmen
- A 321 x 399 grid of pixel values
  - Row and Column = position
- A 1 x 128079 vector
  - "Unraveling" the image

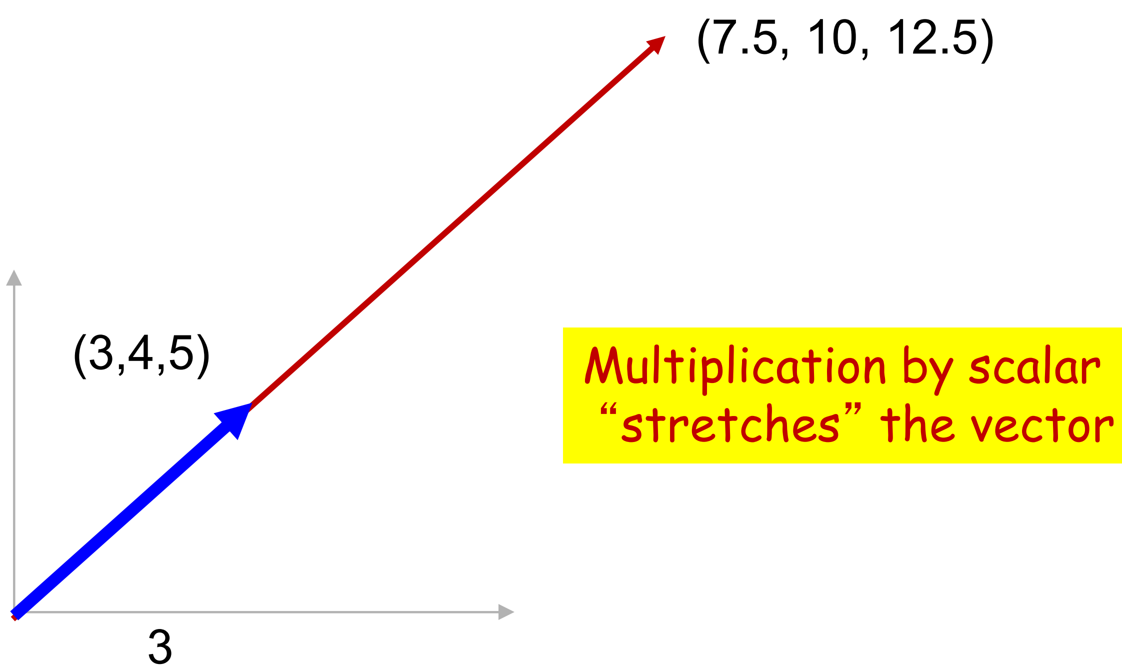
$$[1 \quad 1 \quad . \quad 1 \quad 1 \quad . \quad 0 \quad 0 \quad 0 \quad . \quad . \quad 1]$$

  - Note: This can be recast as the grid that forms the image

Vector operations

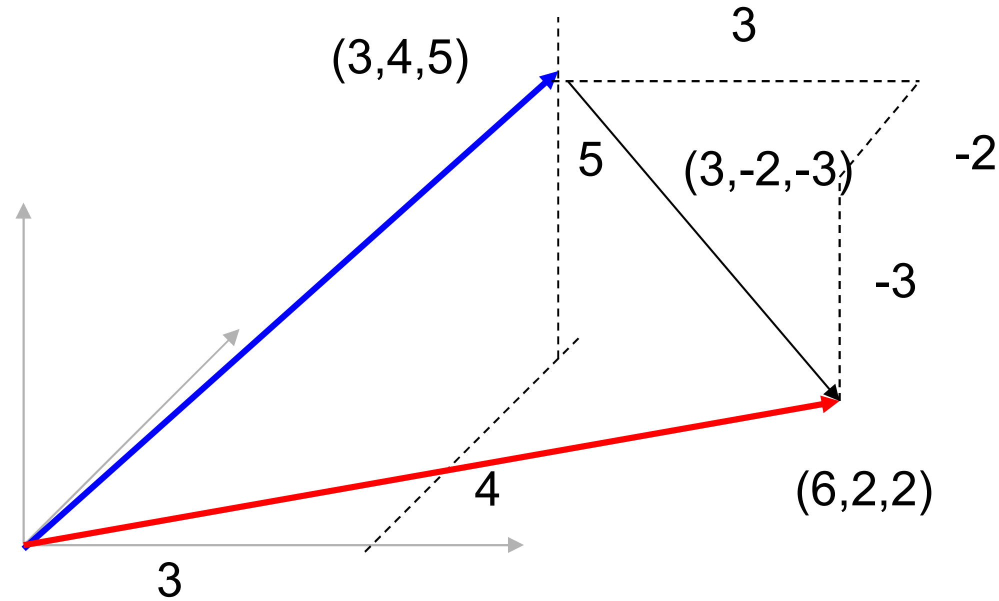
- Addition
- Multiplication
- Inner product
- Outer product

Vector Operations: Multiplication by scalar



- Vector multiplication by scalar: each component multiplied by scalar
  - 2.5 x[3,4,5] = [7.5, 10, 12.5]
- Note: as a result, vector norm is also multiplied by the scalar
  - ||2.5 x [3,4,5]|| = 2.5x|| [3, 4, 5]||

Vector Operations: Addition



- Vector addition: individual components add
  - [3,4,5] + [3,-2,-3] = [6,2,2]

Vector operation: Inner product

- Multiplication of a row vector by a column vector to result in a scalar
  - Note order of operation
  - The *inner* product between two row vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is the product of $\boldsymbol{u}^T$ and $\boldsymbol{v}$
  - Also called the “dot” product

$$\boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

$$\boldsymbol{u}.\boldsymbol{v} = \boldsymbol{u}^T\boldsymbol{v} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = a.d + b.e + c.f$$

Vector operation: Inner product

- The inner product of a vector with itself is its squared norm
  - This will be the squared length

$$\boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

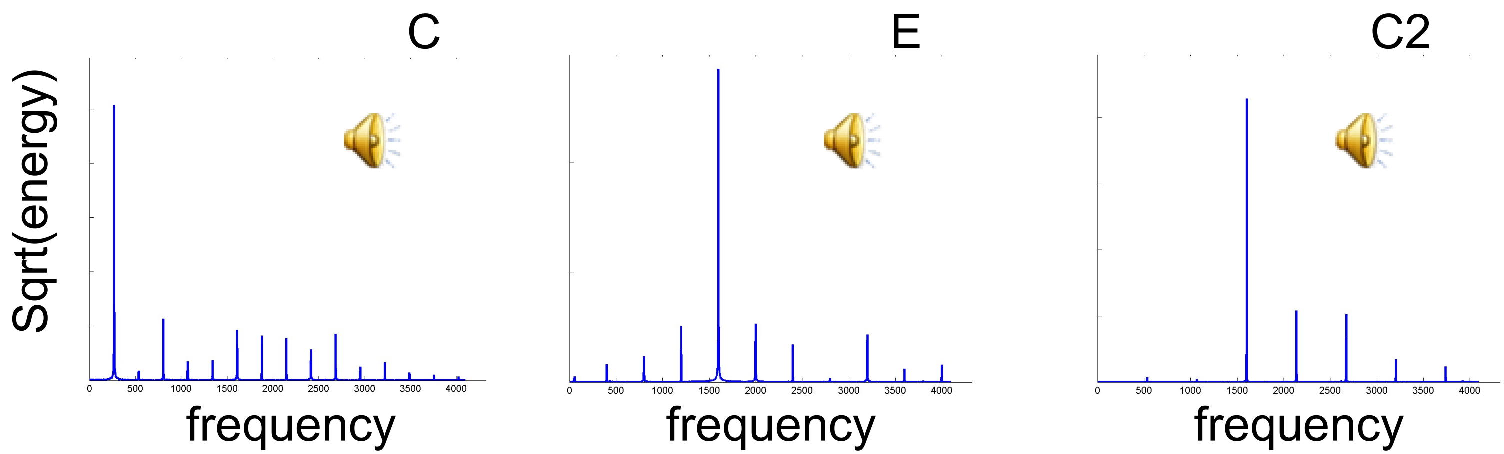
$$\boldsymbol{u}.\boldsymbol{u} = \boldsymbol{u}^T\boldsymbol{u} = a^2 + b^2 + c^2 = \|\boldsymbol{u}\|^2$$

Vector dot product

- Example:
  - Coordinates are yards, not ave/st
  - $\mathbf{a} = [200\ 1600]$,
    $\mathbf{b} = [770\ 300]$
- The dot product of the two vectors relates to the length of a *projection*
  - How much of the first vector have we covered by following the second one?
  - Must normalize by the length of the "target" vector

$$\frac{\mathbf{a}\cdot\mathbf{b}^T}{\|\mathbf{a}\|} = \frac{\begin{bmatrix}200 & 1600\end{bmatrix}\cdot\begin{bmatrix}770\\300\end{bmatrix}}{\left\|\begin{bmatrix}200 & 1600\end{bmatrix}\right\|} \approx 393\text{yd}$$

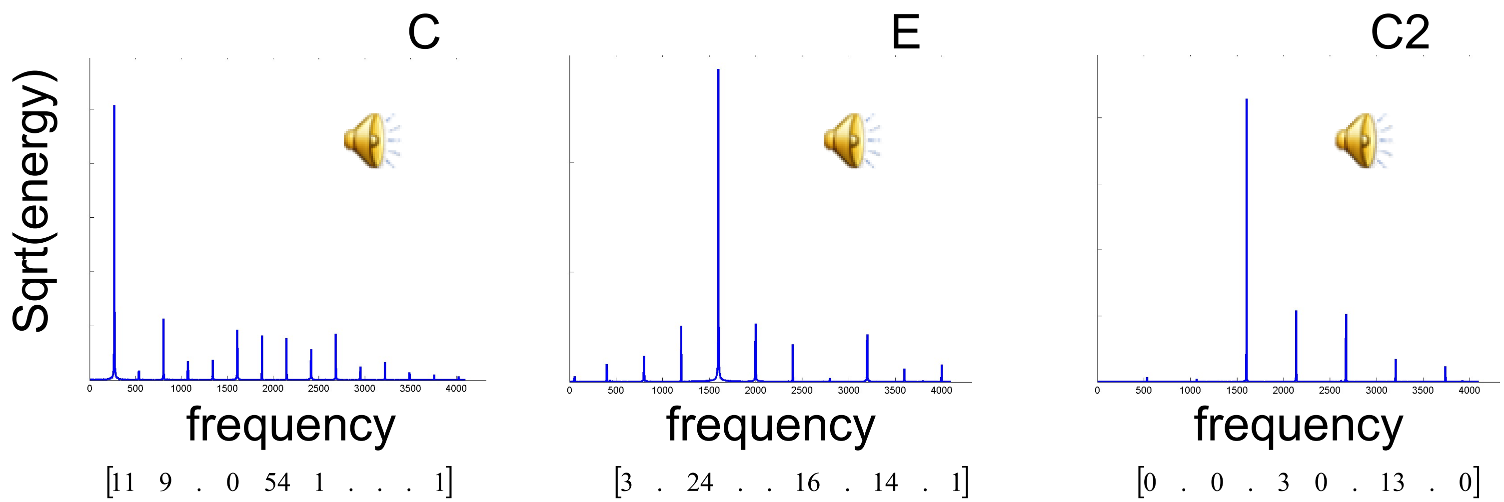
Vector dot product



[11 9 . 0 54 1 . . . 1] [3 . 24 . . 16 . 14 . 1] [0 . 0 . 3 0 . 13 . 0]

- Vectors are spectra
  - Energy at a discrete set of frequencies
  - Actually 1 x 4096
  - X axis is the *index* of the number in the vector
    - Represents frequency
  - Y axis is the value of the number in the vector
    - Represents magnitude

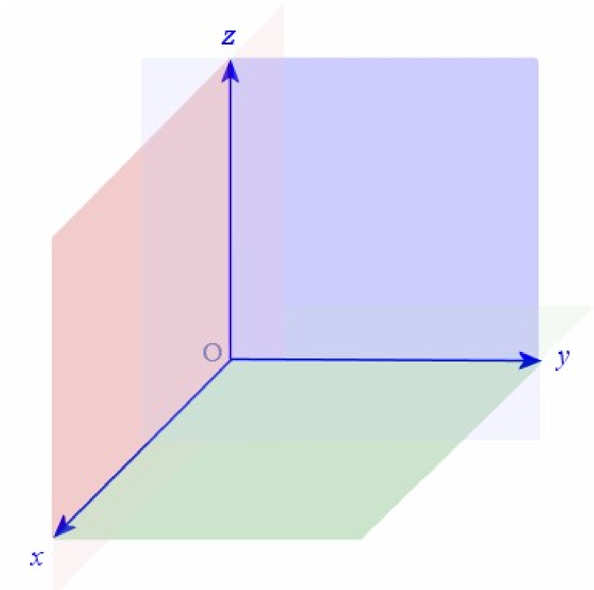
Vector dot product



- How much of C is also in E
  - How much can you fake a C by playing an E
  - C.E / |C||E| = 0.1
  - Not very much
- How much of C is in C2?
  - C.C2 / |C| /|C2| = 0.5
  - Not bad, you can fake it

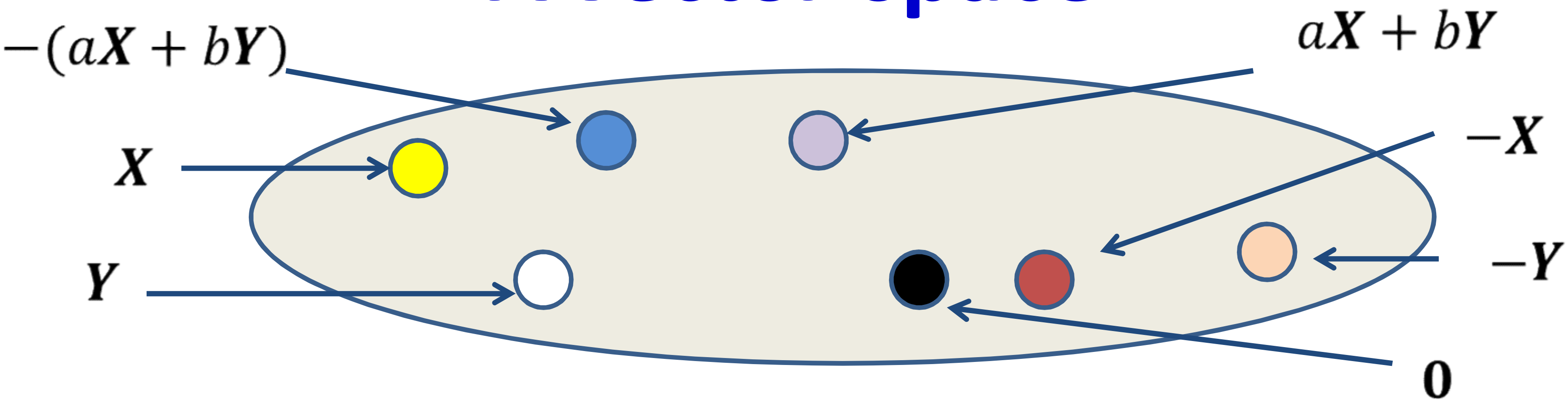
The notion of a “Vector Space”

An introduction to *spaces*



- Conventional notion of "space": a geometric construct of a certain number of "dimensions"
  - E.g. the 3-D space that this room and every object in it lives in

A *vector* space



- A *vector space* is an infinitely large set of vectors with the following properties
  - The set includes the zero vector (of all zeros)
  - The set is "closed" under addition
    - If X and Y are in the set, $aX + bY$ is also in the set for any two scalars a and b
  - For every X in the set, the set also includes the additive inverse $Y = -X$, such that $X + Y = 0$

Additional Properties

- Additional requirements:
  - Scalar multiplicative identity element exists: $1X = X$
  - Addition is associative: $X + Y = Y + X$
  - Addition is commutative: $(X+Y)+Z = X+(Y+Z)$
  - Scalar multiplication is commutative: $a(bX) = (ab)\,X$
  - Scalar multiplication is distributive: $(a+b)X = aX + bX$ $a(X+Y) = aX + aY$

Example of vector space

$$\mathbf{S} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \; for\, all\; x, y, z \;\in\; \mathcal{R} \right\}$$

- Set of *all* three-component column vectors
  - Note we used the term three-*component,* rather than three-*dimensional*
- The set includes the zero vector
- For every $\mathbf{X}$ in the set $\alpha \in \mathcal{R}$, every $\alpha\mathbf{X}$ is in the set
- For every $\mathbf{X}, \mathbf{Y}$ in the set, $\alpha\mathbf{X} + \beta\mathbf{Y}$ is in the set
- $-\mathbf{X}$ is in the set
- Etc.

Example: a function space

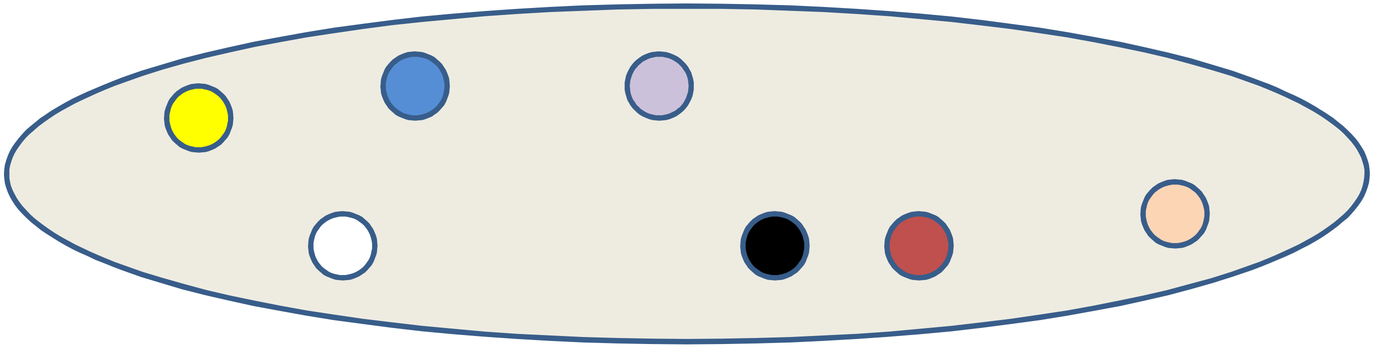
$$\mathbf{S} = \left\{ \begin{matrix} a\cos(\mathrm{x}) + b\sin(3\mathrm{x}) \;\; for\ all\ a, b, \in \mathcal{R}\}, \\ \mathrm{x} \in [-\pi, \pi] \end{matrix} \right\}$$

- Entries are *functions* from $[-\pi, \pi]$ to $[-1,1]$

$$f: [-\pi, \pi] \longrightarrow [-1,1]$$

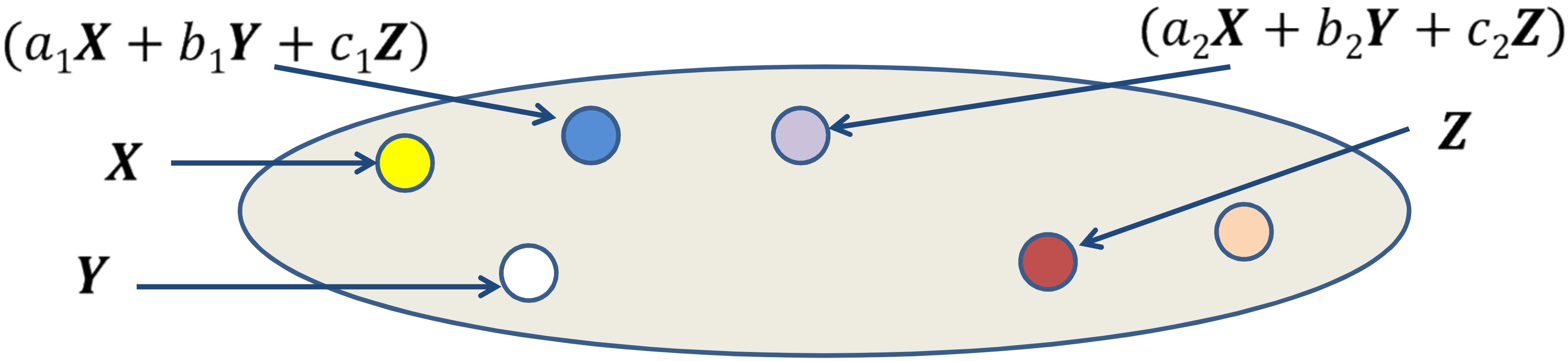
- Define $(f+g)(\mathrm{x}) = f(\mathrm{x}) + g(\mathrm{x})$ for any f and g in the set
- Verify that this is a space!

Dimension of a space



- Every element in the space can be composed of linear combinations of some other elements in the space
  - For any $\mathbf{X}$ in $\mathbf{S}$ we can write $\mathbf{X} = a\mathbf{Y}_1 + b\mathbf{Y}_2 + c\mathbf{Y}_3..$ for some other $\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$ .. in $\mathbf{S}$
    - Trivial to prove..

Dimension of a space



- What is the smallest subset of elements that can compose the entire set?
  - There may be multiple such sets
- The elements in this set are called “bases”
  - The set is a “basis” set
- The number of elements in the set is the “dimensionality” of the space

Dimensions: Example

$$\mathbf{S} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \; for\; all\; x, y, z \in \mathcal{R} \right\}$$

- What is the dimensionality of this vector space

Dimensions: Example

$$\mathbf{Z} = \left\{ a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, for\ all\ a, b\ \in\ \mathcal{R} \right\}$$

- What is the dimensionality of this vector space?
  - First confirm this is a proper vector space
- Note: all elements in $\mathbf{Z}$ are also in $\mathbf{S}$ (slide 36)
  - $\mathbf{Z}$ is a *subspace* of $\mathbf{S}$

Dimensions: Example

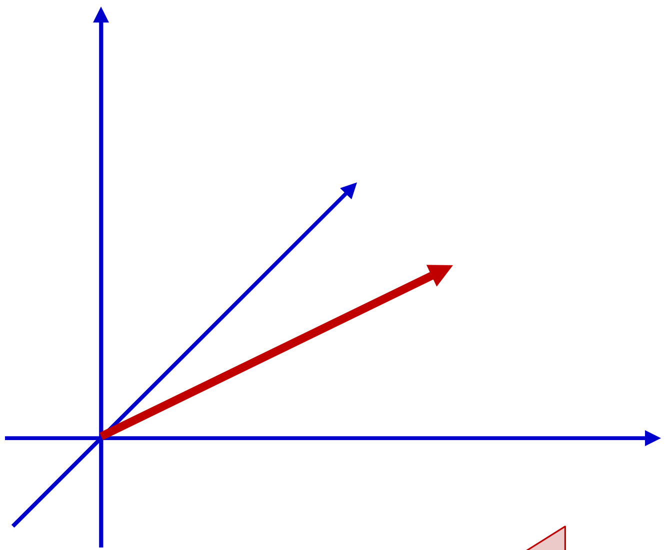
$$\mathbf{S} = \left\{ \begin{matrix} a\cos(\mathrm{x}) + b\sin(3\mathrm{x})\ for\ all\ a, b, \in \ \mathcal{R}\}, \\ \mathrm{x} \in [-\pi, \pi] \end{matrix} \right\}$$

- What is the dimensionality of this space?

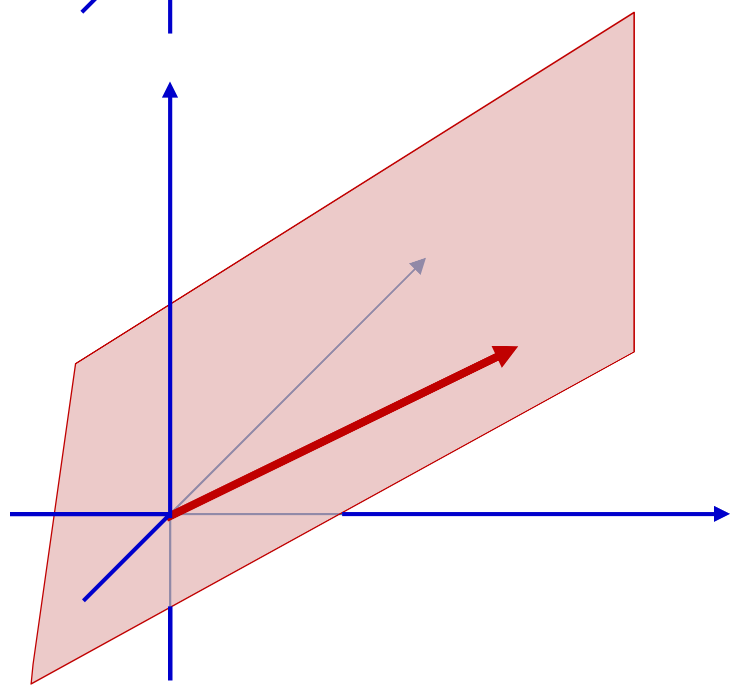
- Return to reality..

Returning to dimensions..

- Two interpretations of "dimension"
- The *spatial* dimension of a vector:
  - The number of components in the vector
  - An N-component vector "lives" in an N-dimensional space
  - Essentially a "stand-alone" definition of a vector against "standard" bases



- The *embedding* dimension of the vector
  - The minimum number of bases required to specify the vector
  - The dimensionality of the *subspace* the vector actually lives in
  - Only makes sense in the context where the vector is one element of a restricted set, e.g. a subspace or hyperplane



- Much of machine learning and signal processing is aimed at finding the latter from collections of vectors

Matrices..

What is a *matrix*

A 2x3 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2.2 & 6 \\ 3.1 & 1 & 5 \end{bmatrix}$$

A 3x2 matrix

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

- Rectangular (or square) arrangement of numbers

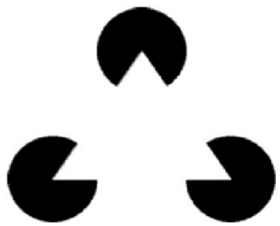
Dimensions of a matrix

- The matrix size is specified by the number of rows and columns

$$\mathbf{c} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \ \mathbf{r} = \begin{bmatrix} a & b & c \end{bmatrix}$$

  - c = 3x1 matrix: 3 rows and 1 column (vectors are matrices too)
  - r = 1x3 matrix: 1 row and 3 columns

$$\mathbf{S} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ \mathbf{R} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$



  - S = 2 x 2 matrix
  - R = 2 x 3 matrix
  - Pacman = 321 x 399 matrix

Dimensionality and Transposition

- A transposed matrix gets all its row (or column) vectors transposed in order
  - An NxM matrix becomes an MxN matrix

$$\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \mathbf{x}^T = \begin{bmatrix} a & b & c \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} a & b & c \end{bmatrix}, \quad \mathbf{y}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \quad \mathbf{X}^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

$\mathbf{M} =$  $, \quad \mathbf{M}^T =$ 

What is a *matrix*

A 2x3 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2.2 & 6 \\ 3.1 & 1 & 5 \end{bmatrix}$$

A 3x2 matrix

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

- A matrix by itself is uninformative, except through its relationship to vectors

Interpreting matrices

- Matrices as transforms
- Matrices as data containers
- Matrices as compositional building blocks for vector spaces

Interpreting matrices

- Matrices as transforms
- Matrices as data containers
- Matrices as compositional building blocks for vector spaces

Matrices as transforms

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

- Multiplying a vector by a matrix *transforms* the vector
  - $\mathbf{Ab} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{12}b_3 + a_{14}b_4 \\ a_{21}b_1 + a_{22}b_2 + a_{32}b_3 + a_{44}b_4 \\ a_{31}b_1 + a_{32}b_2 + a_{32}b_3 + a_{44}b_4 \end{bmatrix}$
- A matrix is a *transform* that *transforms* a vector
  - Above example: *left multiplication*. Matrix transforms a column vector
  - Dimensions must match!!
    - No. of columns of matrix = size of vector
    - Result inherits the number of rows from the matrix

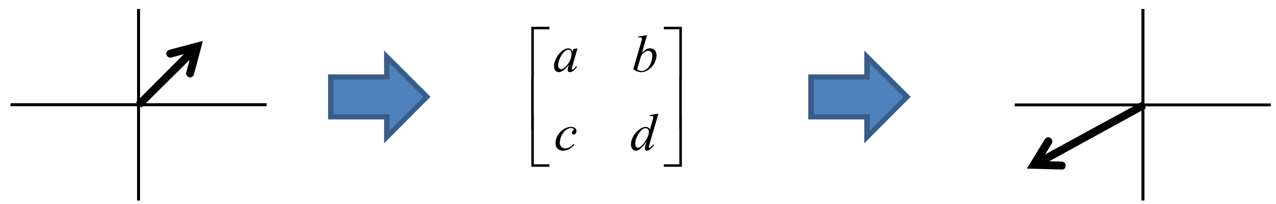
Matrices as transforms

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

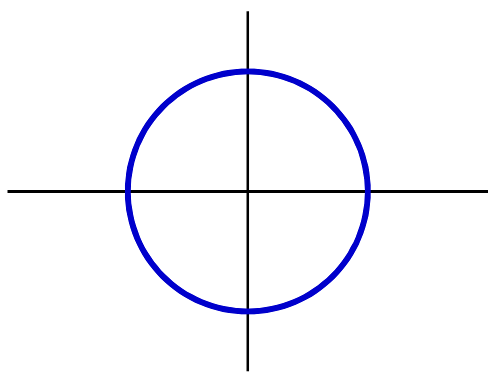
- Multiplying a vector by a matrix *transforms* the vector
  - $\mathbf{bA} = [b_1 \quad b_2 \quad b_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{21}b_2 + a_{31}b_3 \\ a_{12}b_1 + a_{22}b_2 + a_{32}b_3 \\ a_{13}b_1 + a_{23}b_2 + a_{33}b_3 \\ a_{14}b_1 + a_{24}b_2 + a_{34}b_3 \end{bmatrix}^T$
- A matrix is a *transform* that *transforms* a vector
  - Example: *right multiplication.* Matrix transforms a row vector
  - Dimensions must match!!
    - No. of rows of matrix = size of vector
    - Result inherits the number of columns from the matrix

Matrices transform a space

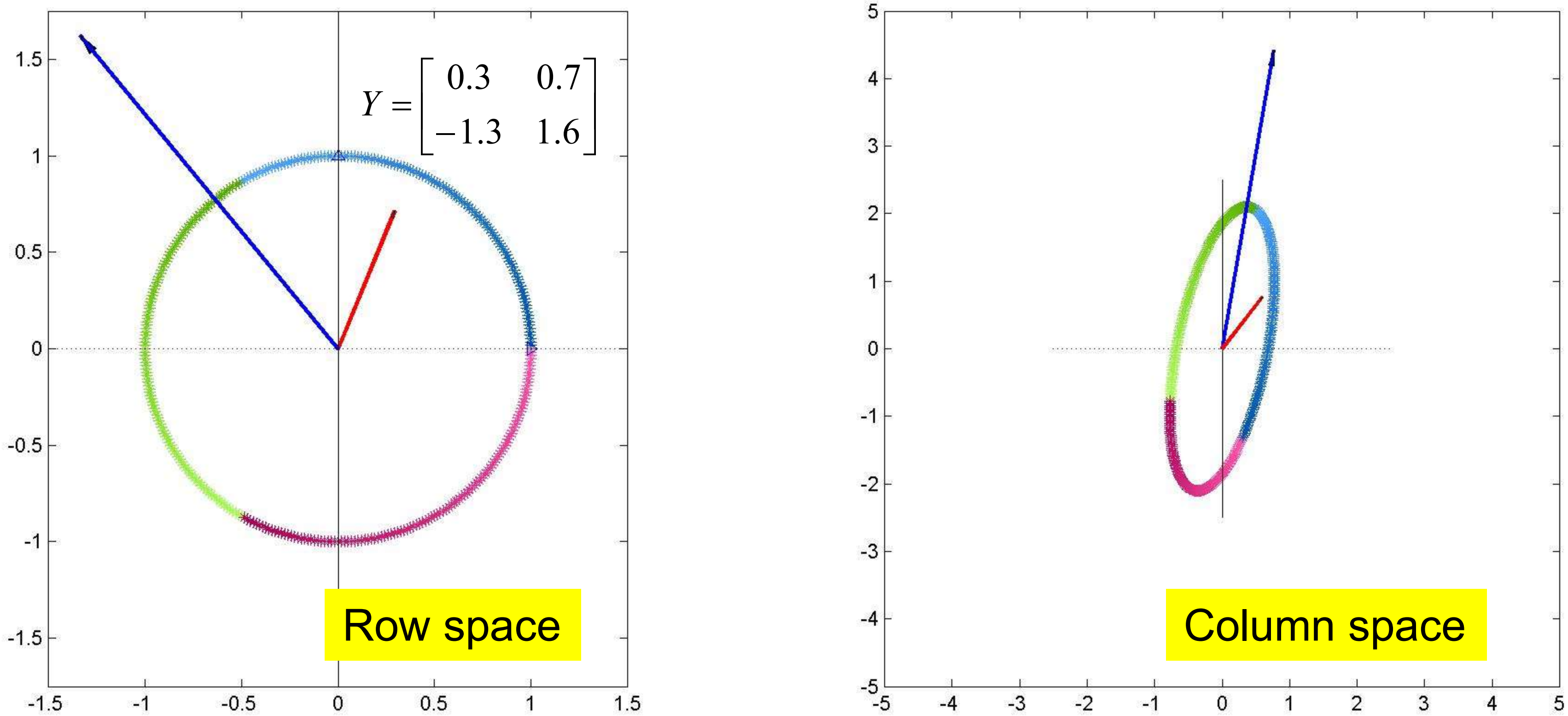
- A matrix is a ***transform*** that modifies vectors and vector spaces



- So how does it transform the *entire space*?
- E.g. how will it transform the following figure?

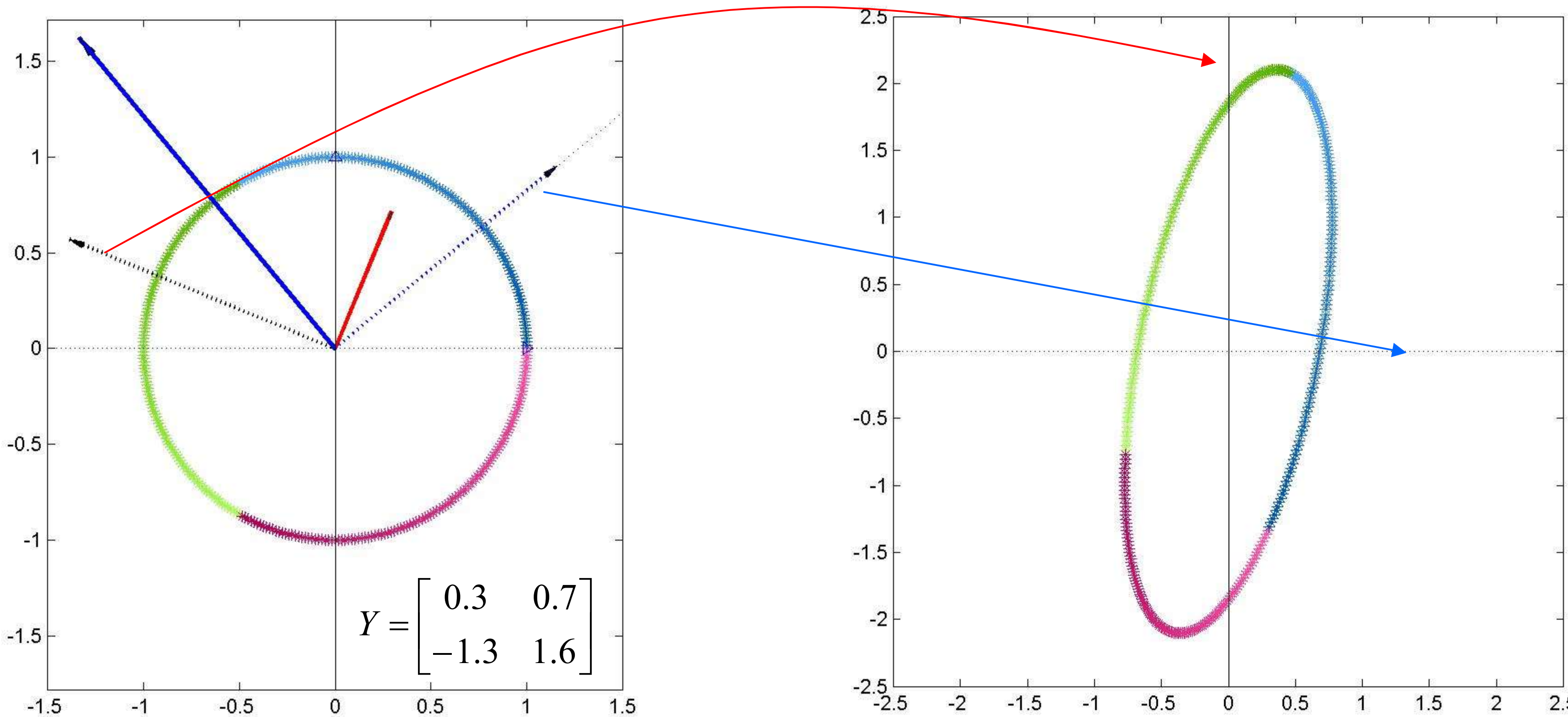


Multiplication of vector space by matrix



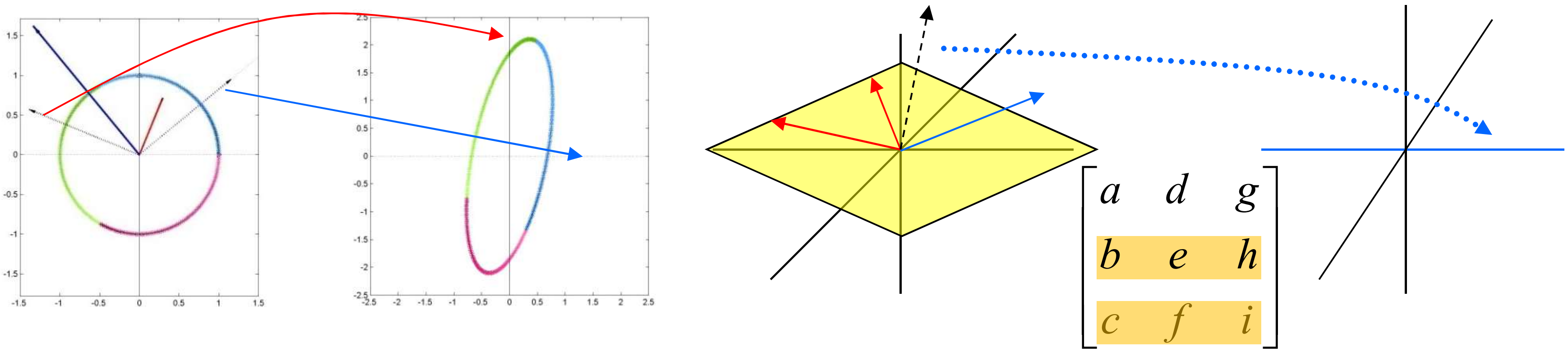
- The matrix rotates and scales the space
  - Including its own row vectors

Multiplication of vector space by matrix



- The *normals* to the row vectors in the matrix become the new axes
  - X axis = normal to the *second* row vector
    - Scaled by the inverse of the length of the *first* row vector

Matrix Multiplication



- The k-th axis corresponds to the normal to the hyperplane represented by the 1..k-1,k+1..N-th row vectors in the matrix
  - Any set of K-1 vectors represent a hyperplane of dimension K-1 or less

- The distance along the new axis equals the length of the projection on the k-th row vector
  - Expressed in inverse-lengths of the vector

Interpreting matrices

- Matrices as transforms
- Matrices as data containers
- Matrices as compositional building blocks for vector spaces

Matrices as data containers

- A matrix can be vertical stacking of row vectors

$$\mathbf{R} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

  – The space of all vectors that can be composed from the rows of the matrix is the *row space* of the matrix

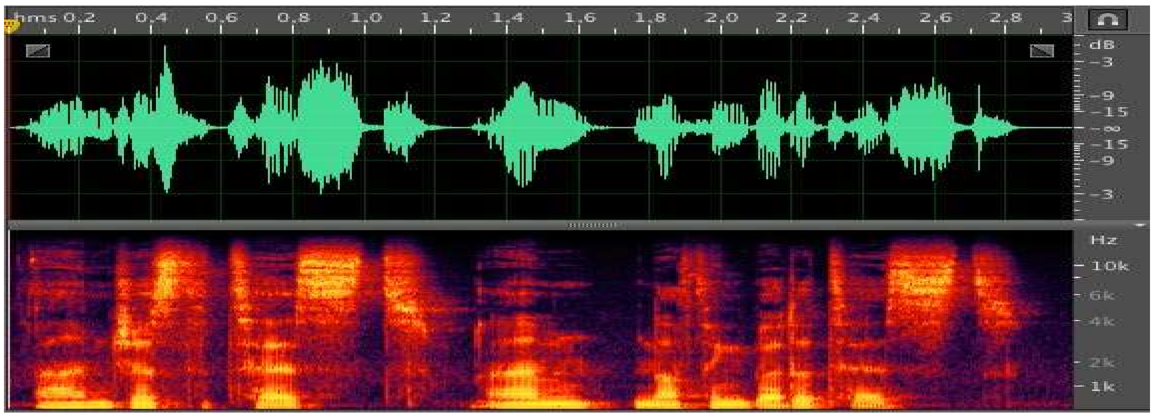
- Or a horizontal arrangement of column vectors

$$\mathbf{R} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

  – The space of all vectors that can be composed from the columns of the matrix is the *column space* of the matrix

Representing a signal as a matrix

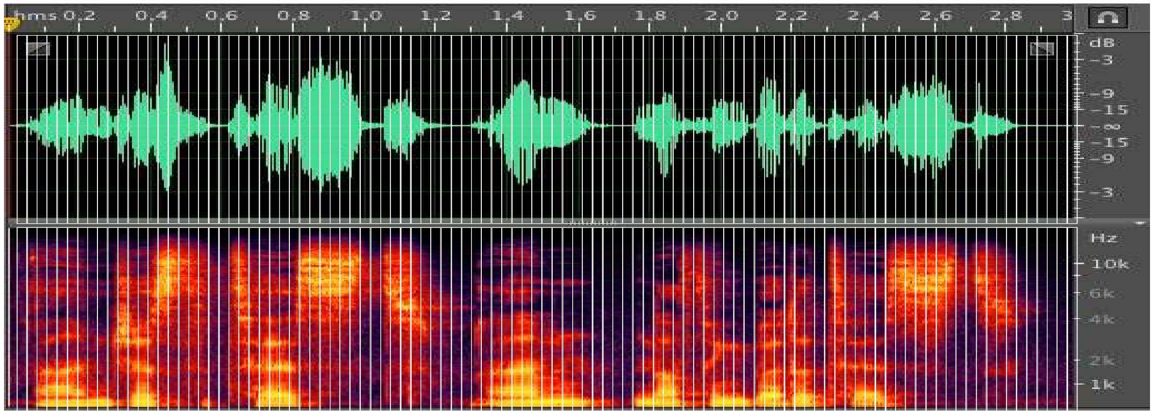
- Time series data like audio signals are often represented as spectrographic matrices



- Each column is the spectrum of a short segment of the audio signal

Representing a signal as a matrix

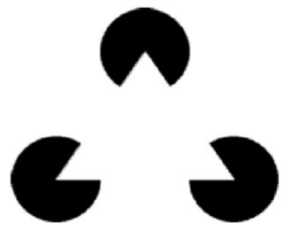
- Time series data like audio signals are often represented as spectrographic matrices



- Each column is the spectrum of a short segment of the audio signal

Representing a signal as a matrix

- Images are often just represented as matrices

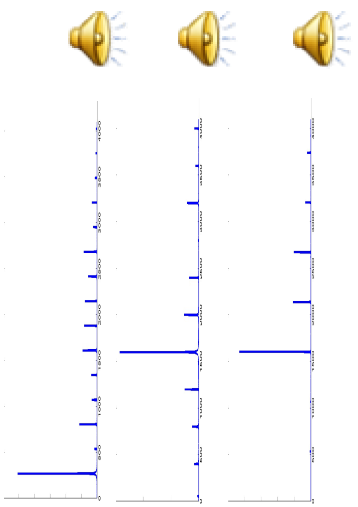


```
>> X(1:32:end,1:40:end)

ans =

     1     1     1     1     1     1     1     1     1     1
     1     1     1     1     0     0     0     1     1     1
     1     1     1     1     0     0     0     1     1     1
     1     1     1     1     0     1     0     1     1     1
     1     1     1     1     1     1     1     1     1     1
     1     1     1     1     1     1     1     1     1     1
     1     1     0     1     1     1     1     1     0     1
     1     0     0     1     1     1     1     1     0     0
     1     0     0     0     1     1     1     0     0     0
     1     0     0     0     1     1     1     0     0     0
     1     1     1     1     1     1     1     1     1     1
```

Storing collections of data



X

$$\begin{bmatrix} 1 & 3 & 0 \\ . & . & 0 \\ 9 & 24 & . \\ . & . & 1 \end{bmatrix}$$

$$\begin{bmatrix} f_{11} & \cdots & f_{K1} \\ \vdots & \ddots & \vdots \\ f_{1N} & \cdots & f_{KN} \end{bmatrix}$$

- Individual data instances can be packed into columns (or rows) of a matrix
  - A "data container" matrix

Interpreting matrices

- Matrices as transforms
- Matrices as data containers
- Matrices as compositional building blocks for vector spaces

Matrices as space constructors

- Right multiplying a matrix by a column vector mixes the columns of the matrix according to the numbers in the vector

  - $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{32} & a_{32} & a_{33} & a_{34} \end{bmatrix}$ $\qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$

$$\boldsymbol{Ab} = b_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + b_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + b_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} + b_4 \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix}$$

- "Mixes" the columns
  - "Transforms" row space to column space
- "Generates" the space of vectors that can be formed by mixing its own columns

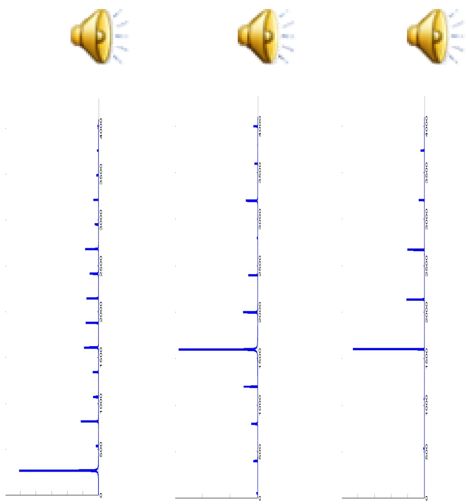
Multiplying a vector by a matrix

- Left multiplying a matrix by a row vector mixes the rows of the matrix according to the numbers in the vector
  - $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{32} & a_{32} & a_{33} & a_{34} \end{bmatrix}$ $\quad\quad \boldsymbol{b} = [b_1 \quad b_2 \quad b_3]$

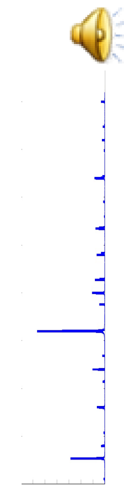
$$\boldsymbol{bA} = b_1[a_{11} \quad a_{12} \quad a_{13} \quad a_{14}] + b_2[a_{21} \quad a_{22} \quad a_{23} \quad a_{24}]$$
$$+ b_3[a_{31} \quad a_{32} \quad a_{33} \quad a_{34}]$$

- "Mixes" the rows
  - "Transforms" column space to row space
- "Generates" the space of vectors that can be formed by mixing its own rows

Matrix multiplication: Mixing vectors

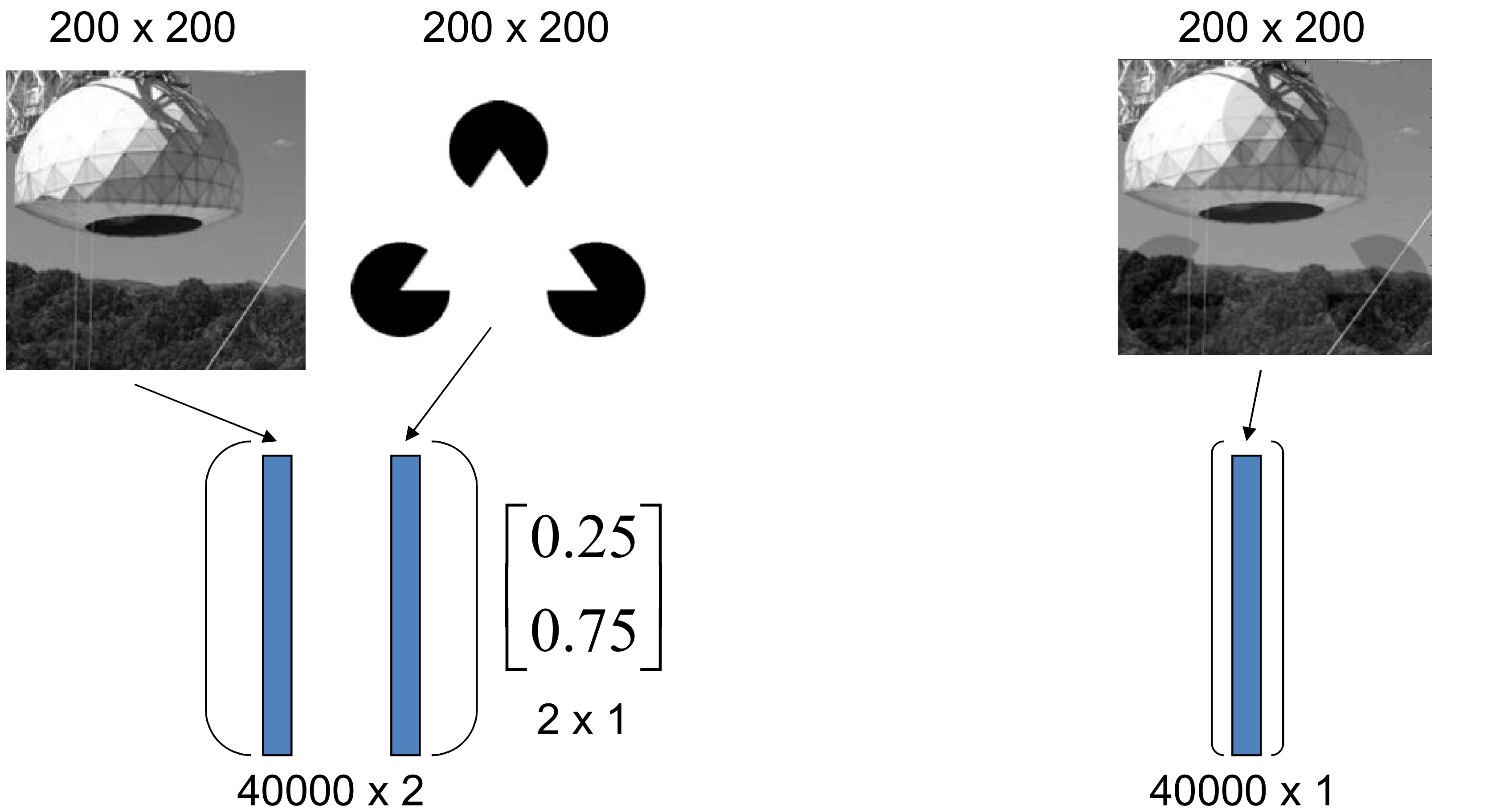


$$\overset{X}{\begin{bmatrix} 1 & 3 & 0 \\ . & . & 0 \\ 9 & 24 & . \\ . & . & 1 \end{bmatrix}} \overset{Y}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}} = \begin{bmatrix} 7 \\ . \\ . \\ 2 \end{bmatrix}$$



- A physical example
  - The three column vectors of the matrix X are the spectra of three notes
  - The multiplying column vector Y is just a mixing vector
  - The result is a sound that is the mixture of the three notes

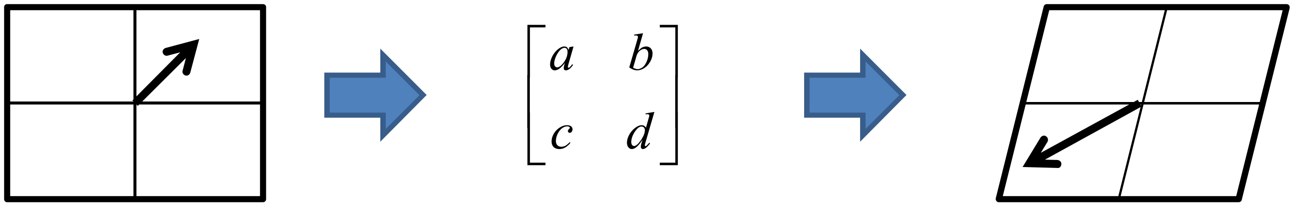
Matrix multiplication: Mixing vectors



- Mixing two images
  - The images are arranged as columns
    - position value not included
  - The result of the multiplication is rearranged as an image

Interpretations of a matrix

- As a ***transform*** that modifies vectors and vector spaces



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- As a ***container*** for data (vectors)

$$\begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \end{bmatrix}$$

- As a ***generator*** of vector spaces..

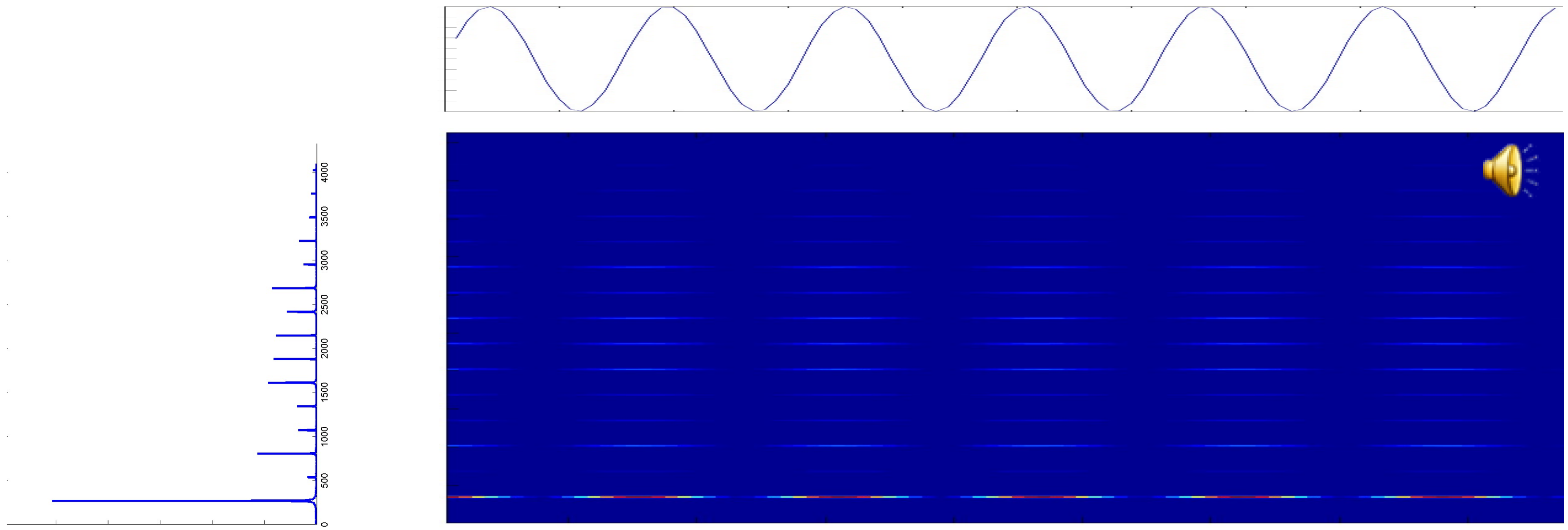
Matrix ops..

Vector multiplication: Outer product

- Product of a column vector by a row vector
- Also called vector *direct* product
- Results in a *matrix*
- *Transform or collection of vectors?*

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d & e & f \end{bmatrix} = \begin{bmatrix} a \cdot d & a \cdot e & a \cdot f \\ b \cdot d & b \cdot e & b \cdot f \\ c \cdot d & c \cdot e & c \cdot f \end{bmatrix}$$

Vector outer product



- The column vector is the spectrum
- The row vector is an amplitude modulation
- The outer product is a spectrogram
  - Shows how the energy in each frequency varies with time
  - The pattern in each column is a scaled version of the spectrum
  - Each row is a scaled version of the modulation

Matrix multiplication

$$\begin{bmatrix} a_{11} & . & . & a_{1N} \\ a_{21} & . & . & a_{2N} \\ . & . & . & . \\ a_{M1} & . & . & a_{MN} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & . & b_{1K} \\ . & . & . \\ b_{N1} & . & b_{NK} \end{bmatrix} = \begin{bmatrix} \sum_j a_{1j} b_{j1} & . & . & \sum_j a_{1j} b_{jK} \\ . & . & . & . \\ . & . & . & . \\ \sum_j a_{Mj} b_{j1} & . & . & \sum_j a_{Mj} b_{jK} \end{bmatrix}$$

- Standard formula for matrix multiplication

Matrix multiplication

$$\begin{bmatrix} a_{11} & . & . & a_{1N} \\ a_{21} & . & . & a_{2N} \\ . & . & . & . \\ a_{M1} & . & . & a_{MN} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & . & b_{1K} \\ . & . & . \\ b_{N1} & . & b_{NK} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & . & . & \mathbf{a}_1 \cdot \mathbf{b}_K \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & . & . & \mathbf{a}_2 \cdot \mathbf{b}_K \\ . & . & . & . \\ \mathbf{a}_M \cdot \mathbf{b}_1 & . & . & \mathbf{a}_M \cdot \mathbf{b}_K \end{bmatrix}$$

- Matrix $\boldsymbol{A}$: A column of row vectors
- Matrix $\boldsymbol{B}$: A row of column vectors
- $\boldsymbol{AB}$: A matrix of inner products
  - Mimics the vector outer product rule

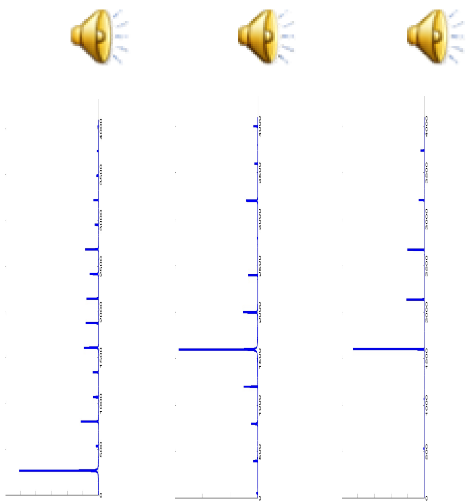
Matrix multiplication: another view

$$\begin{bmatrix} a_{11} & . & . & a_{1N} \\ a_{21} & . & . & a_{2N} \\ . & . & . & . \\ a_{M1} & . & . & a_{MN} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & . & b_{NK} \\ . & . & . \\ b_{N1} & . & b_{NK} \end{bmatrix} = \begin{bmatrix} a_{11} \\ . \\ . \\ a_{M1} \end{bmatrix} \begin{bmatrix} b_{11} & . & b_{1K} \end{bmatrix} + \begin{bmatrix} a_{12} \\ . \\ . \\ a_{M2} \end{bmatrix} \begin{bmatrix} b_{21} & . & b_{2K} \end{bmatrix} + \ldots + \begin{bmatrix} a_{1N} \\ . \\ . \\ a_{MN} \end{bmatrix} \begin{bmatrix} b_{N1} & . & b_{NK} \end{bmatrix}$$

- The outer product of the first column of A and the first row of B + outer product of the second column of A and the second row of B + ….

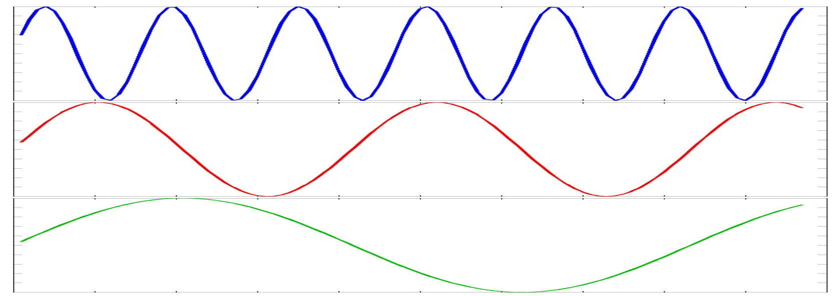
- *Sum of outer products*

Why is that useful?



$$X = \begin{bmatrix} 1 & 3 & 0 \\ . & . & 0 \\ 9 & 24 & . \\ . & . & 1 \end{bmatrix}$$

X

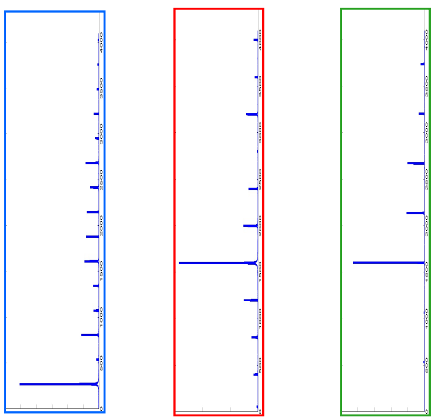


$$Y = \begin{bmatrix} 0 & 0.5 & 0.75 & 1 & 0.75 & 0.5 & 0 & . & . & . & . & . \\ 1 & 0.9 & 0.7 & 0.5 & 0 & 0.5 & . & . & . & . & . & . \\ 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 0.95 & 1 & . & . & . & . & . \end{bmatrix}$$

Y

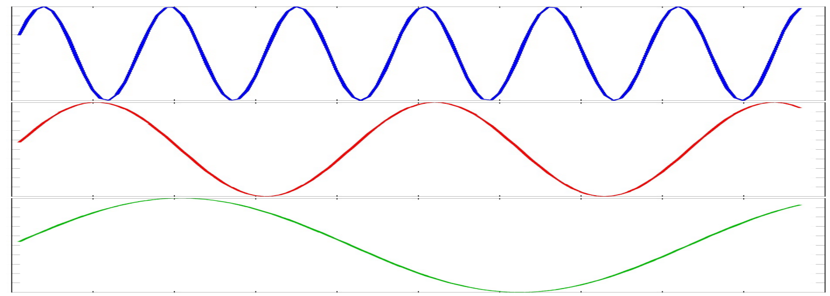
- Sounds: Three notes modulated independently

Matrix multiplication: Mixing modulated spectra



$$X = \begin{bmatrix} 1 & 3 & 0 \\ . & . & 0 \\ 9 & 24 & . \\ . & . & 1 \end{bmatrix}$$

X

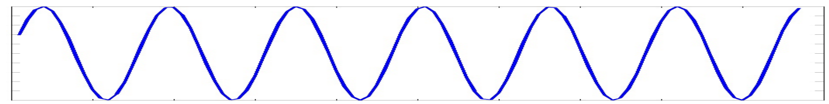


$$Y = \begin{bmatrix} 0 & 0.5 & 0.75 & 1 & 0.75 & 0.5 & 0 & . & . & . & . & . \\ 1 & 0.9 & 0.7 & 0.5 & 0 & 0.5 & . & . & . & . & . & . \\ 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 0.95 & 1 & . & . & . & . & . \end{bmatrix}$$

Y

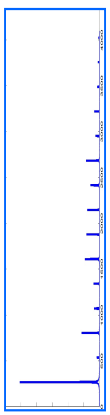
- Sounds: Three notes modulated independently

Matrix multiplication: Mixing modulated spectra



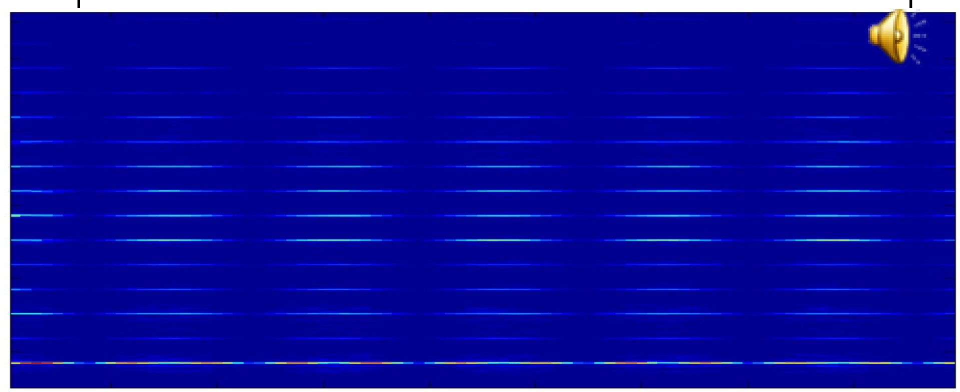
$$\begin{bmatrix} 0 & 0.5 & 0.75 & 1 & 0.75 & 0.5 & 0 & . & . & . & . & . \end{bmatrix}$$

Y



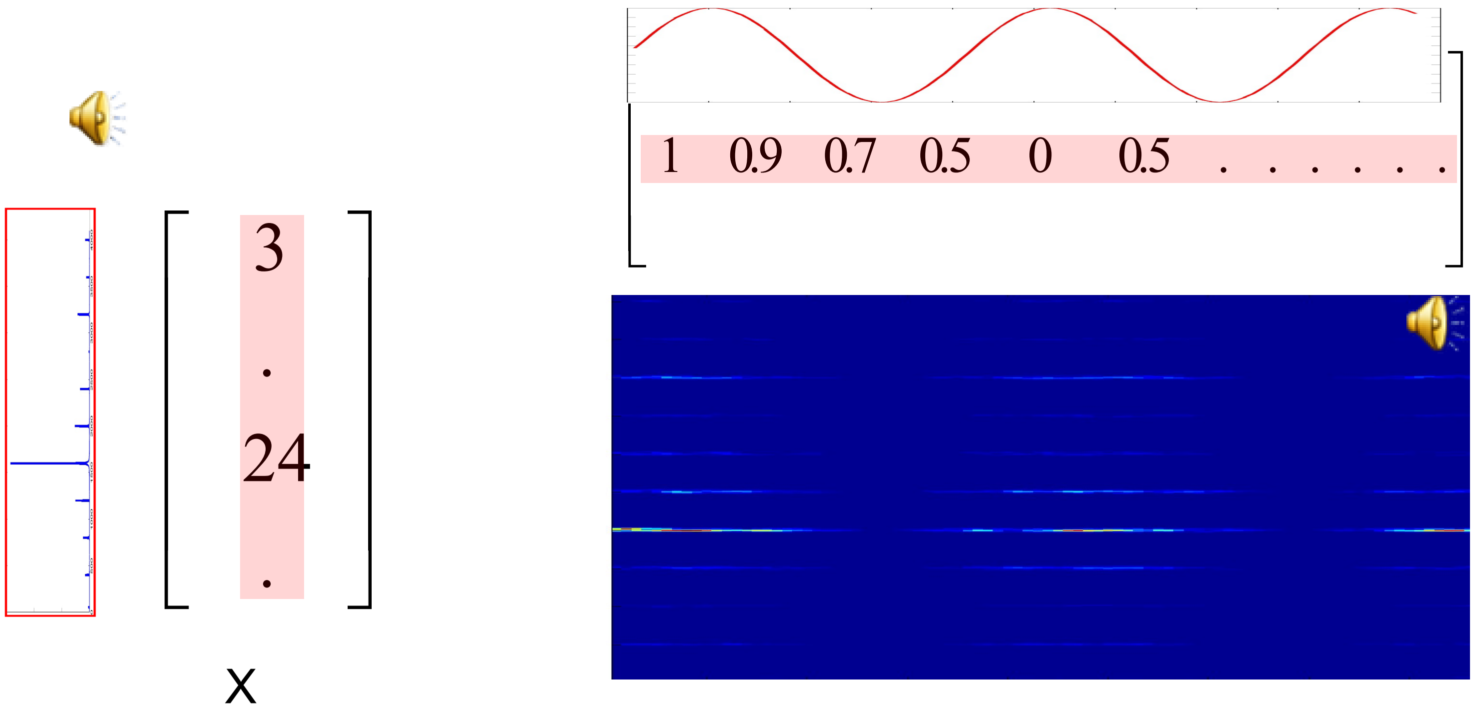
$$\begin{bmatrix} 1 & & \\ . & & \\ 9 & . & \\ . & & \end{bmatrix}$$

X



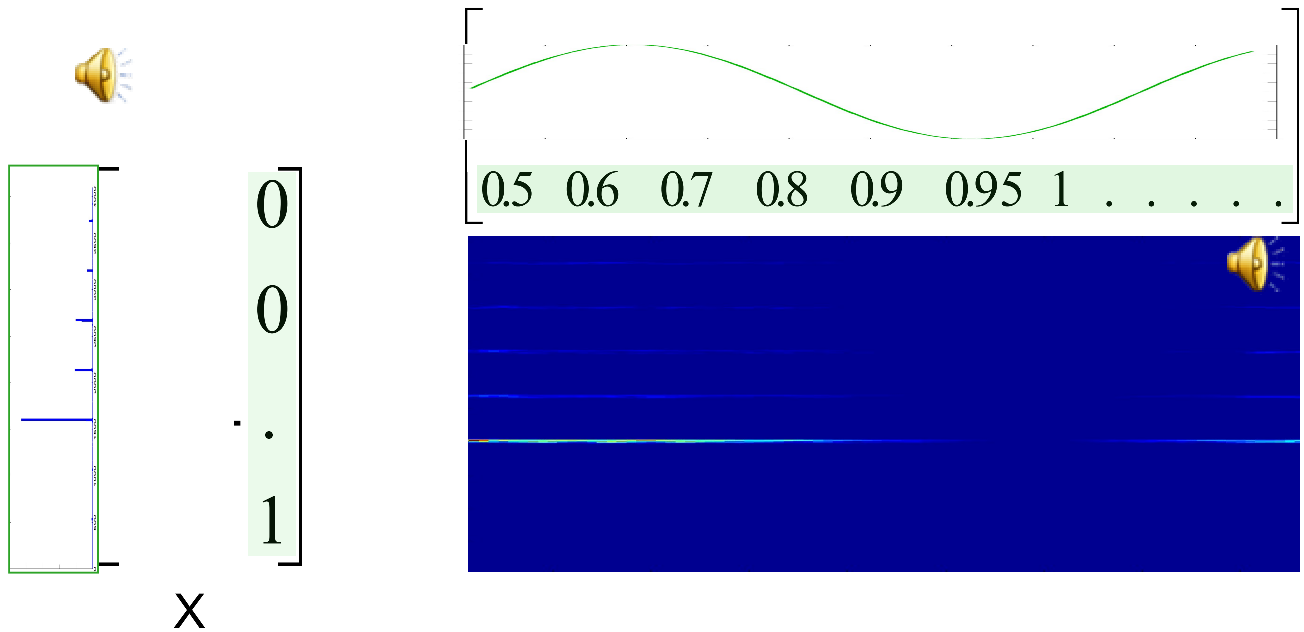
- Sounds: Three notes modulated independently

Matrix multiplication: Mixing modulated spectra



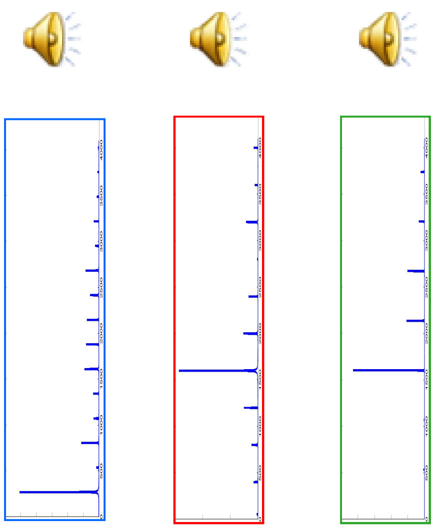
- Sounds: Three notes modulated independently

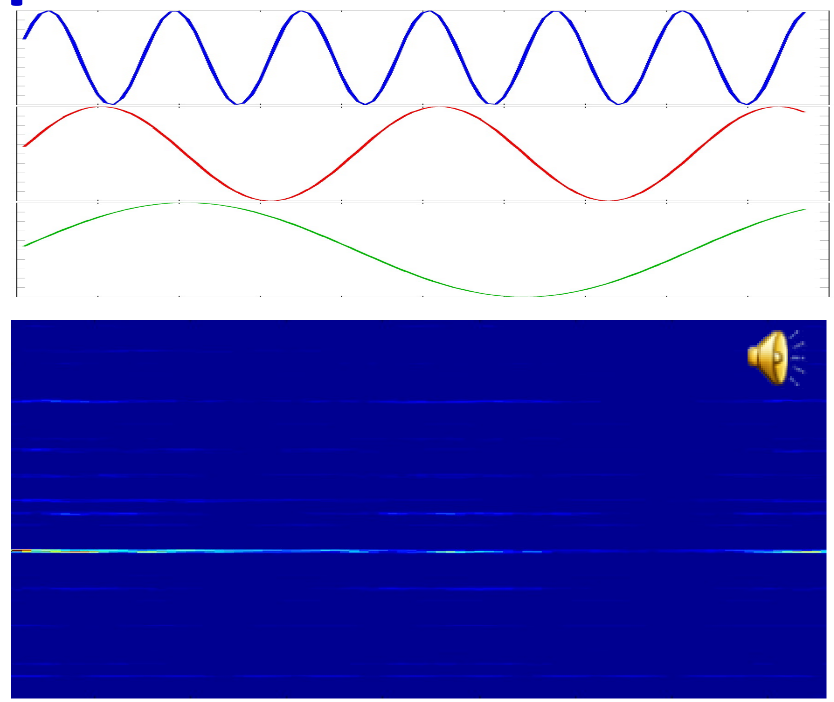
Matrix multiplication: Mixing modulated spectra



- Sounds: Three notes modulated independently

Matrix multiplication: Mixing modulated spectra

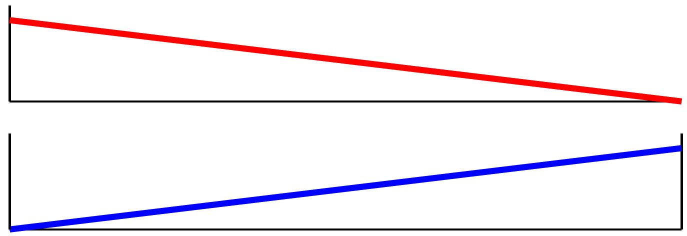




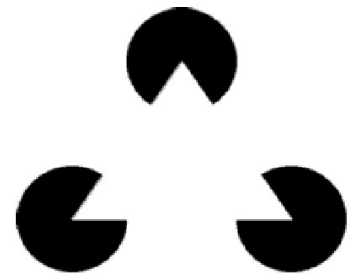
- Sounds: Three notes modulated independently

Matrix multiplication: Image transition

$$\begin{bmatrix} i_1 & j_1 \\ i_2 & j_2 \\ . & . \\ . & . \\ . & . \end{bmatrix}$$

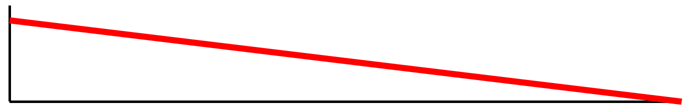


$$\begin{bmatrix} 1 & .9 & .8 & .7 & .6 & .5 & .4 & .3 & .2 & .1 & 0 \\ 0 & .1 & .2 & .3 & .4 & .5 & .6 & .7 & .8 & .9 & 1 \end{bmatrix}$$



- Image1 fades out linearly
- Image 2 fades in linearly

Matrix multiplication: Image transition

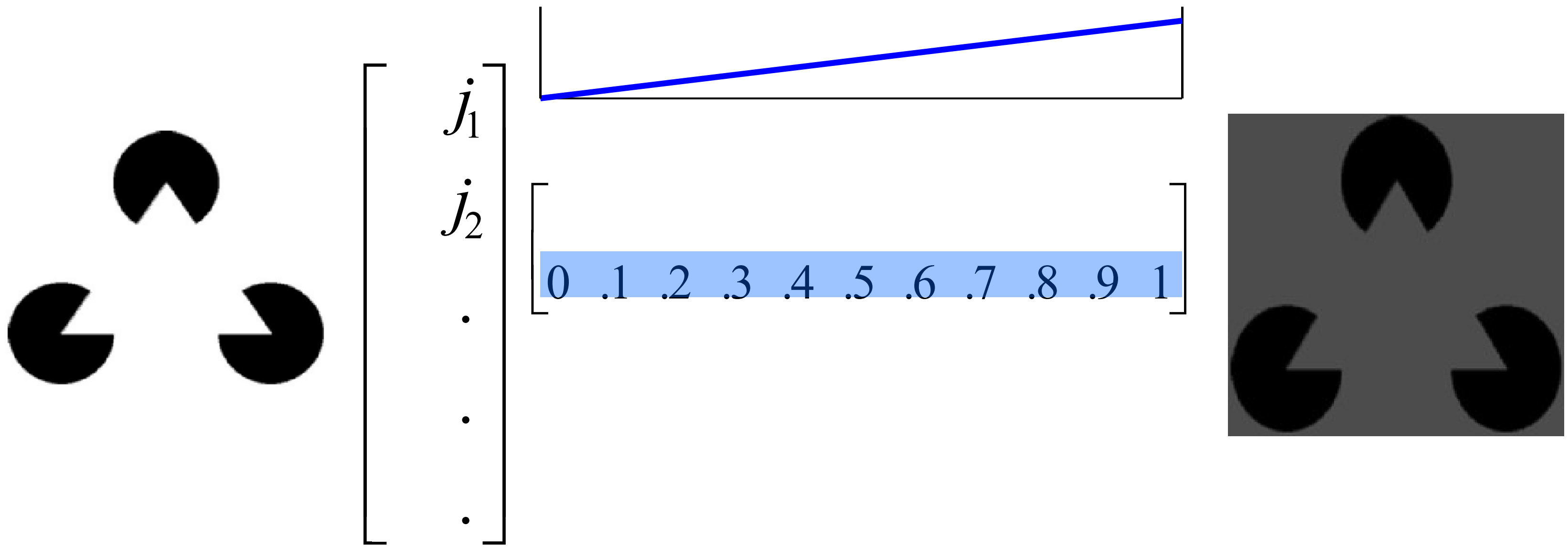


$$\begin{bmatrix} i_1 \\ i_2 \\ . \\ . \\ . \end{bmatrix} \begin{bmatrix} 1 & .9 & .8 & .7 & .6 & .5 & .4 & .3 & .2 & .1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} i_1 & 0.9i_1 & 0.8i_1 & . & . & . & . & . & . & 0 \\ i_2 & 0.9i_2 & 0.8i_2 & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & 0. \\ i_N & 0.9i_N & 0.8i_N & . & . & . & . & . & . & 0 \end{bmatrix}$$

- Each column is one image
  - The columns represent a sequence of images of decreasing intensity
- Image1 fades out linearly

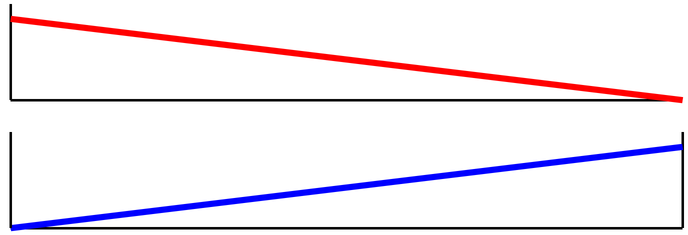
Matrix multiplication: Image transition



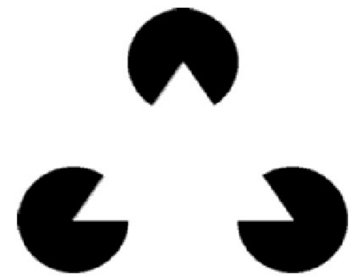
- Image 2 fades in linearly

Matrix multiplication: Image transition

$$\begin{bmatrix} i_1 & j_1 \\ i_2 & j_2 \\ . & . \\ . & . \\ . & . \end{bmatrix}$$



$$\begin{bmatrix} 1 & .9 & .8 & .7 & .6 & .5 & .4 & .3 & .2 & .1 & 0 \\ 0 & .1 & .2 & .3 & .4 & .5 & .6 & .7 & .8 & .9 & 1 \end{bmatrix}$$



- Image1 fades out linearly
- Image 2 fades in linearly

Matrix Operations: Properties

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
  - Actual interpretation: for any vector $\mathbf{x}$
    - $(\mathbf{A} + \mathbf{B})\mathbf{x} = (\mathbf{B} + \mathbf{A})\mathbf{x}$ (column vector $\mathbf{x}$ of the right size)
    - $\mathbf{x}(\mathbf{A} + \mathbf{B}) = \mathbf{x}(\mathbf{B} + \mathbf{A})$ (row vector $\mathbf{x}$ of the appropriate size)

- $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

Multiplication properties

- Properties of vector/matrix products
  - Associative

$$\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C}$$

  - Distributive

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

  - NOT commutative!!!

$$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$$

    - *left multiplications ≠ right multiplications*
  - Transposition

$$(\mathbf{A} \cdot \mathbf{B})^T = \mathbf{B}^T \cdot \mathbf{A}^T$$

The Space of Matrices

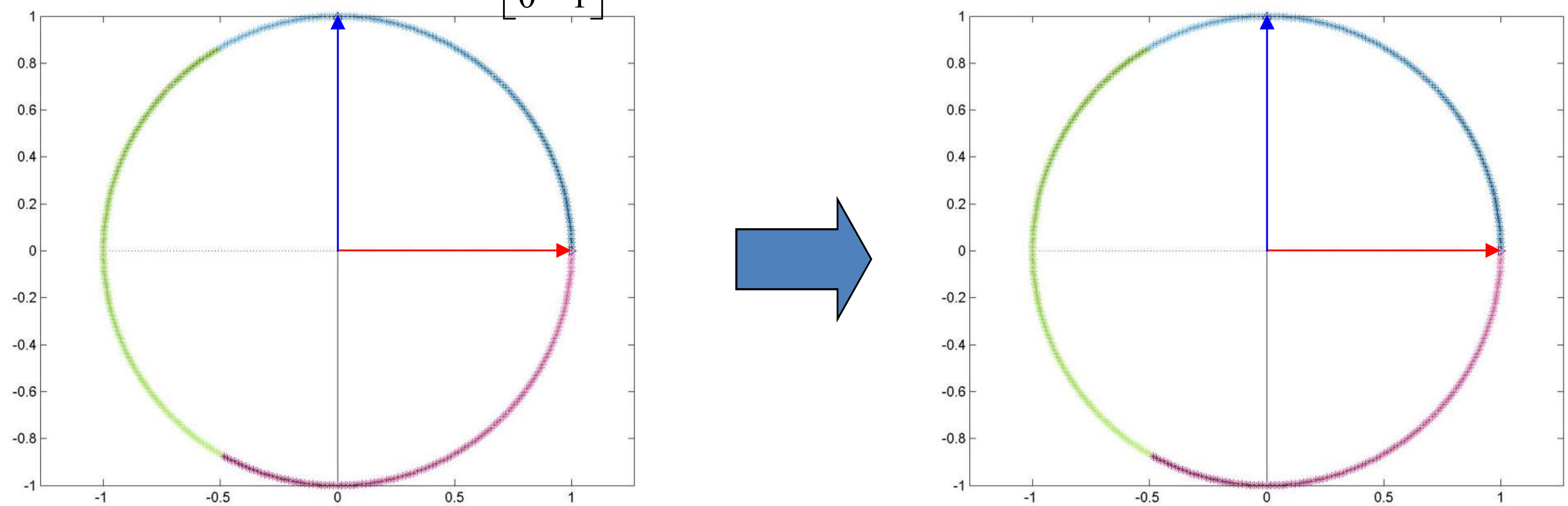
- The set of all matrices of a given size (e.g. all 3x4 matrices) is a space!
  - Addition is closed
  - Scalar multiplication is closed
  - Zero matrix exists
  - Matrices have additive inverses
  - Associativity and commutativity rules apply!

Overview

- Vectors and matrices
- Basic vector/matrix operations
- **Various matrix types**
- Matrix properties
  – Determinant
  – Inverse
  – Rank
- Projections
- Eigen decomposition
- SVD

The Identity Matrix

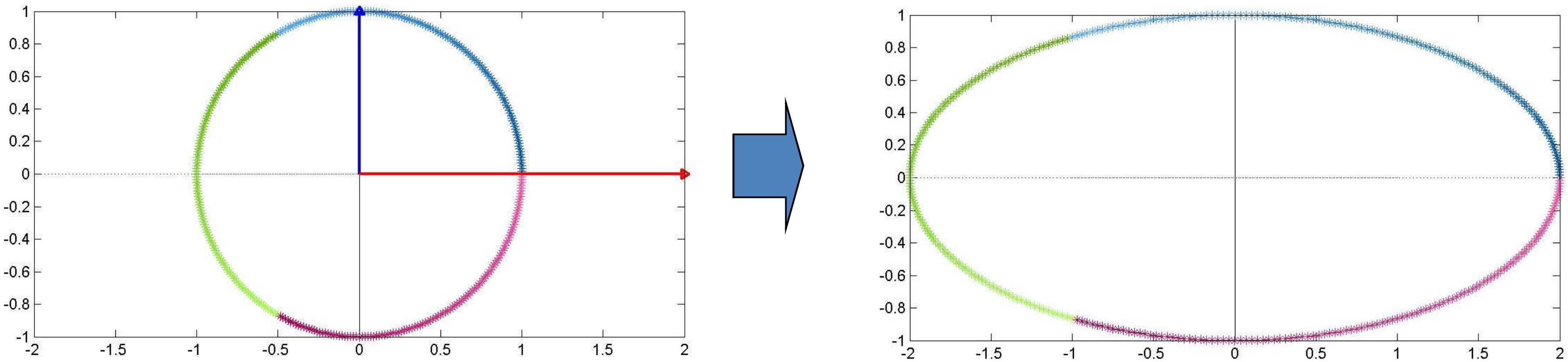
$$Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



- An identity matrix is a square matrix where
  - All diagonal elements are 1.0
  - All off-diagonal elements are 0.0
- Multiplication by an identity matrix does not change vectors

Diagonal Matrix

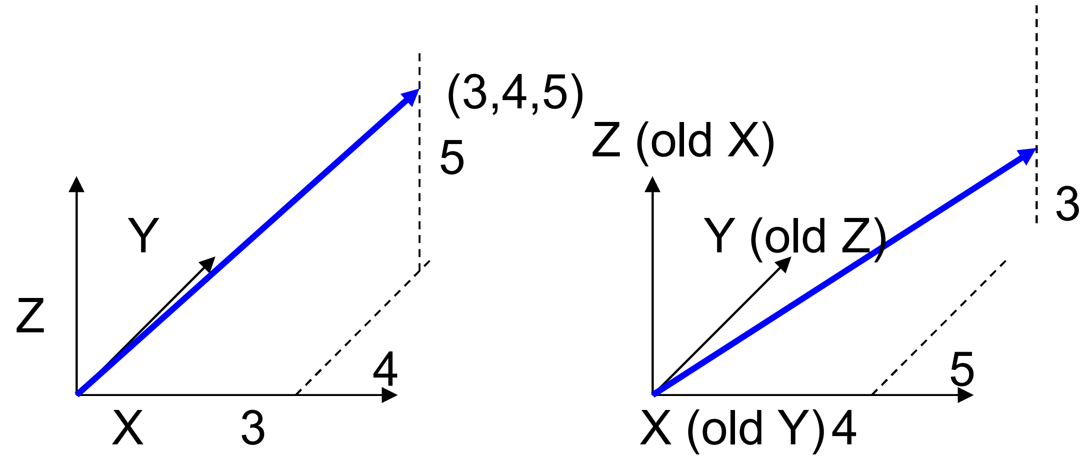
$$Y = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$



- All off-diagonal elements are zero
- Diagonal elements are non-zero
- Scales the axes
  - May flip axes

Permutation Matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$$



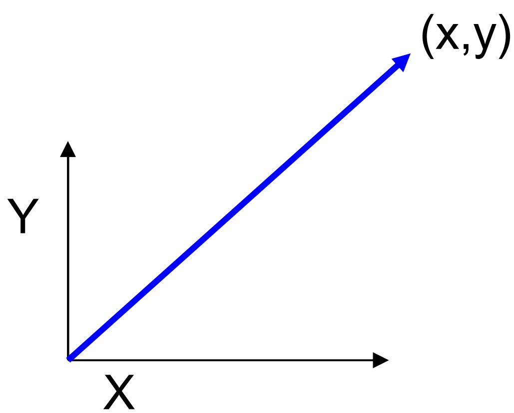
- A permutation matrix simply rearranges the axes
  - The row entries are axis vectors in a different order
  - The result is a combination of rotations and reflections
- The permutation matrix effectively *permutes* the arrangement of the elements in a vector

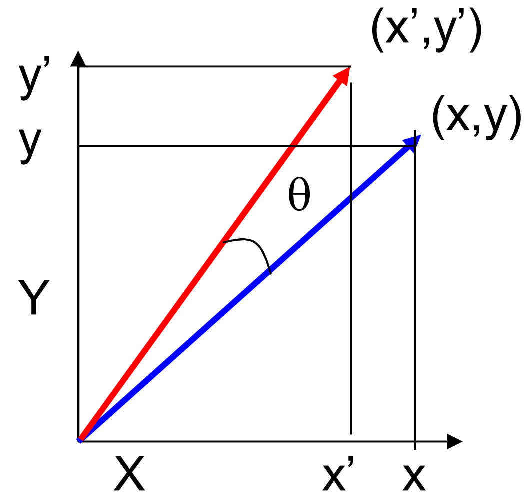
Rotation Matrix

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$X_{new} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

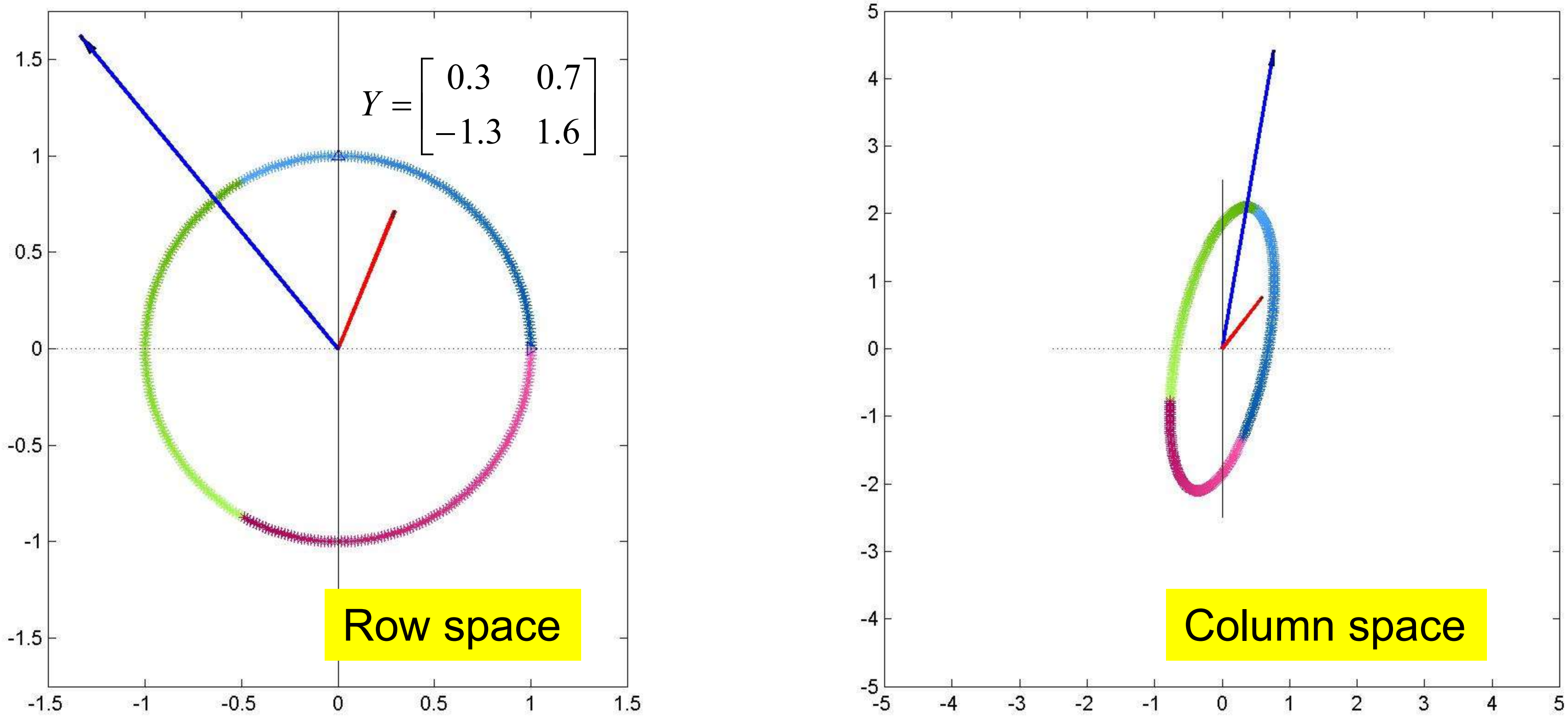
$$R_\theta X = X_{new}$$





- A rotation matrix *rotates* the vector by some angle θ
- Alternately viewed, it rotates the axes
  - The new axes are at an angle θ to the old one

More generally

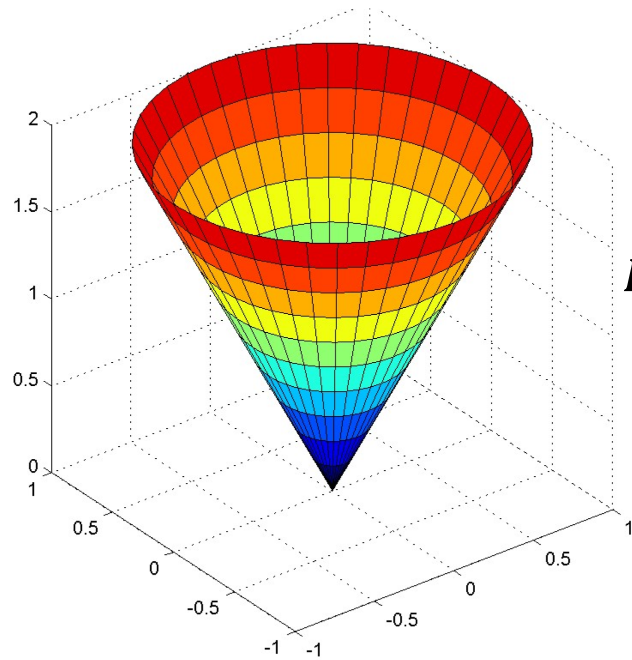


- Matrix operations are combinations of rotations, permutations and stretching

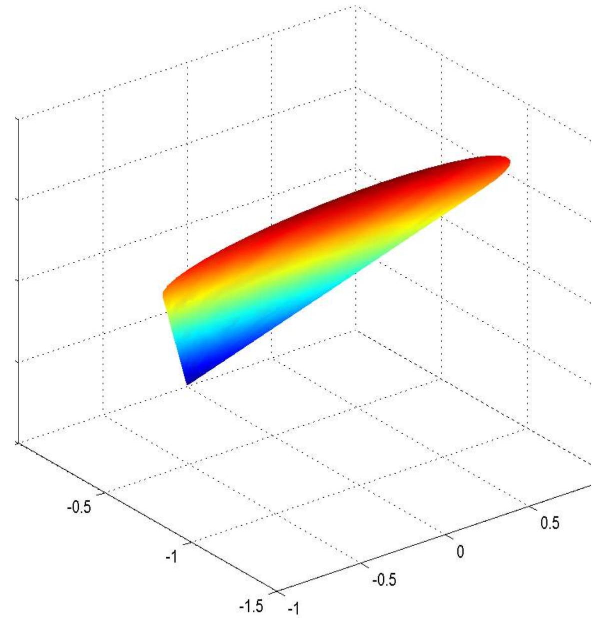
Overview

- Vectors and matrices
- Basic vector/matrix operations
- Various matrix types
- **Matrix properties**
  - Rank
  - Determinant
  - Inverse
- Solving simultaneous equations
- Projections
- Eigen decomposition
- SVD

Matrix Rank and Rank-Deficient Matrices



$\boldsymbol{P} \times Cone =$

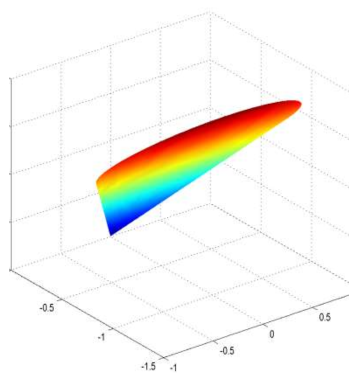


- Some matrices will eliminate one or more dimensions during transformation
  - These are *rank deficient* matrices
  - The **rank** of the matrix is the dimensionality of the transformed version of a **full-dimensional** object

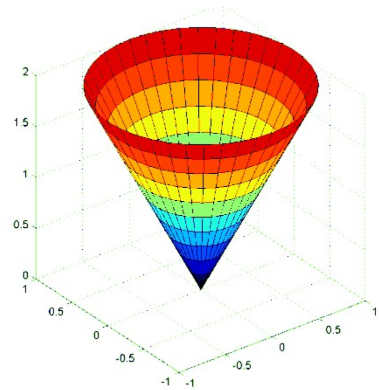
Matrix Rank and Rank-Deficient Matrices

P =

1.0000	0	0
0	0.2500	-0.4330
0	-0.4330	0.7500

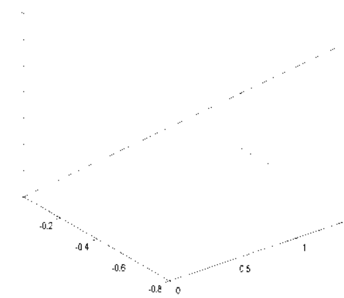


Rank = 2



P2 =

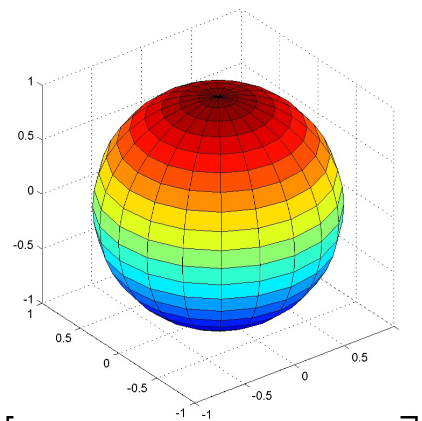
0.5000	-0.2500	0.4330
-0.2500	0.1250	-0.2165
0.4330	-0.2165	0.3750



Rank = 1

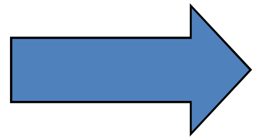
- Some matrices will eliminate one or more dimensions during transformation
  - These are *rank deficient* matrices
  - The rank of the matrix is the dimensionality of the transformed version of a full-dimensional object

Non-square Matrices



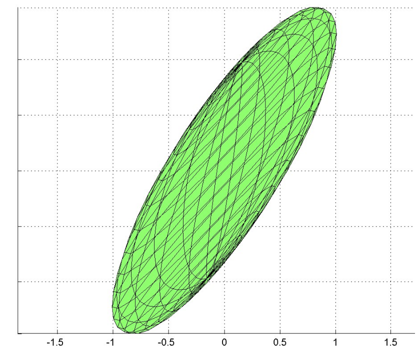
$$\begin{bmatrix} x_1 & x_2 & . & . & x_N \\ y_1 & y_2 & . & . & y_N \\ z_1 & z_2 & . & . & z_N \end{bmatrix}$$

X = 3D data, rank 3



$$\begin{bmatrix} .3 & 1 & .2 \\ .5 & 1 & 1 \end{bmatrix}$$

P = transform

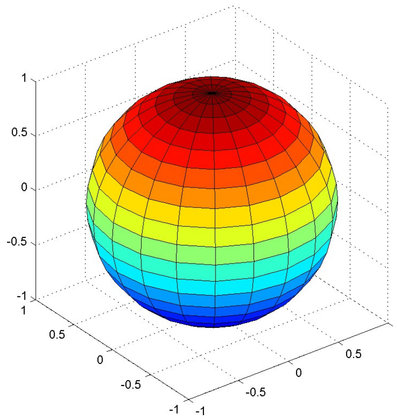


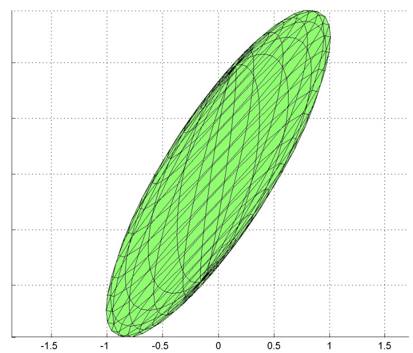
$$\begin{bmatrix} \hat{x}_1 & \hat{x}_2 & . & . & \hat{x}_N \\ \hat{y}_1 & \hat{y}_2 & . & . & \hat{y}_N \end{bmatrix}$$

PX = 2D, rank 2

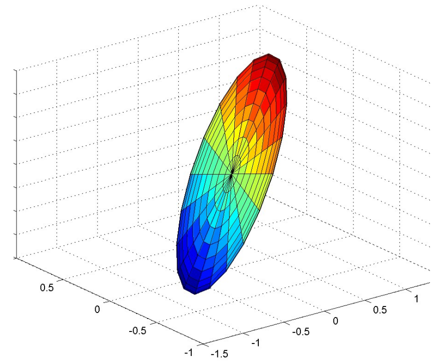
- Non-square matrices add or subtract axes
  - More rows than columns → add axes
    - But does not increase the dimensionality of the data
  - Fewer rows than columns → reduce axes
    - May reduce dimensionality of the data

The Rank of a Matrix





$$\begin{bmatrix} .3 & 1 & .2 \\ .5 & 1 & 1 \end{bmatrix}$$



$$\begin{bmatrix} .8 & .9 \\ .1 & .9 \\ .6 & 0 \end{bmatrix}$$

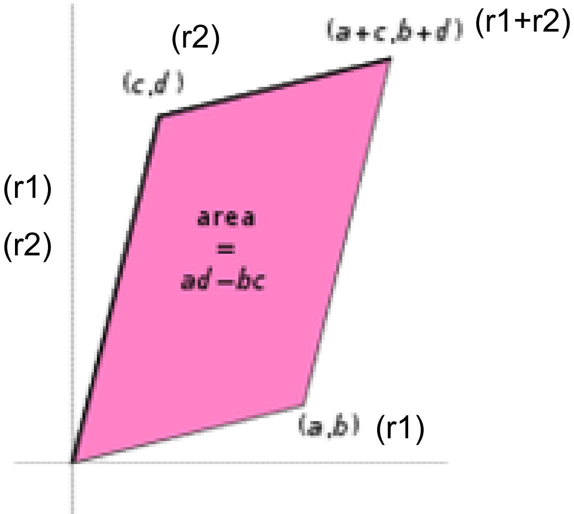
- The matrix rank is the dimensionality of the transformation of a full-dimensioned object in the original space
- The matrix can never *increase* dimensions
  - Cannot convert a circle to a sphere or a line to a circle
- The rank of a matrix can never be greater than the lower of its two dimensions

Rank – an alternate definition

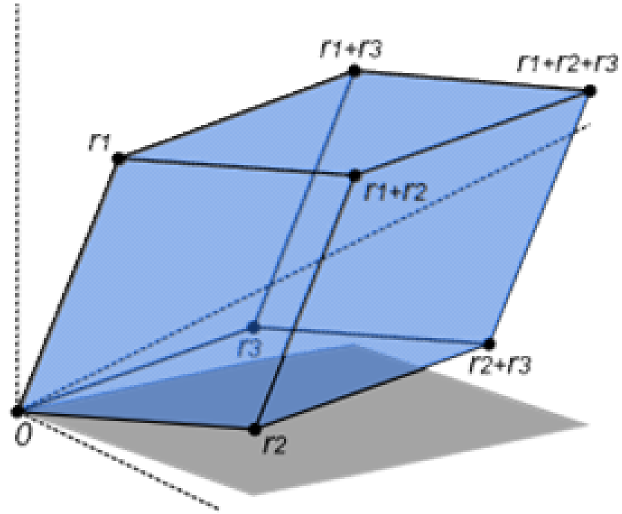
- In terms of bases..
- Will get back to this shortly..

Matrix Determinant

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{matrix} \text{(r1)} \\ \text{(r2)} \end{matrix}$$



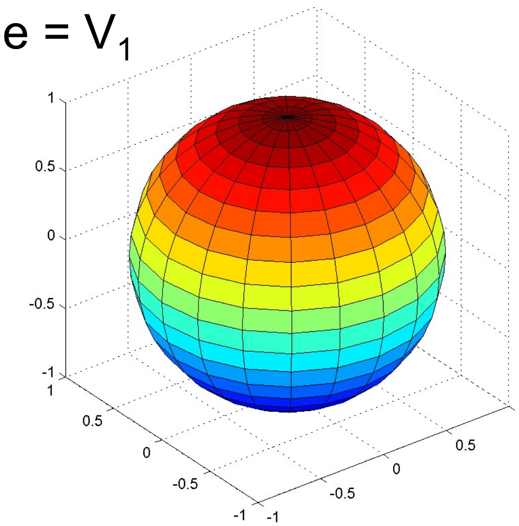
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$



- The determinant is the "volume" of a matrix
- Actually the volume of a parallelepiped formed from its row vectors
  - Also the volume of the parallelepiped formed from its column vectors
- Standard formula for determinant: in text book

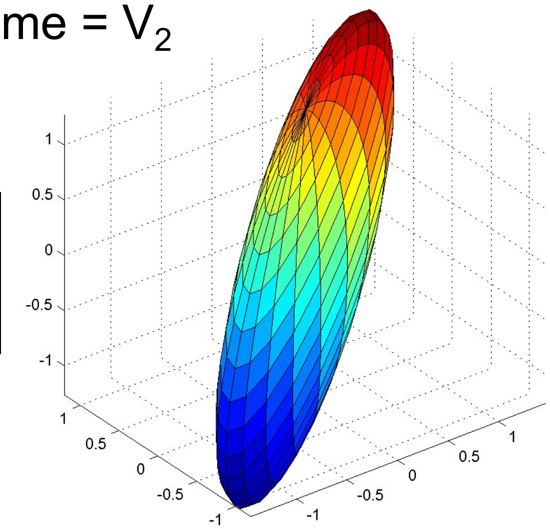
Matrix Determinant: Another Perspective

Volume = V_1



$$\begin{bmatrix} 0.8 & 0 & 0.7 \\ 1.0 & 0.8 & 0.8 \\ 0.7 & 0.9 & 0.7 \end{bmatrix}$$

Volume = V_2



- The (magnitude of the) determinant is the ratio of N-volumes
  - If V_1 is the volume of an N-dimensional sphere "O" in N-dimensional space
    - O is the complete set of points or vertices that specify the object
  - If V_2 is the volume of the N-dimensional ellipsoid specified by A*O, where A is a matrix that transforms the space
  - $|A| = V_2 / V_1$

Matrix Determinants

- Matrix determinants are *only defined for square matrices*
  – They characterize volumes in linearly transformed space of the same dimensionality as the vectors

- Rank deficient matrices have determinant 0
  – Since they compress full-volumed N-dimensional objects into zero-volume N-dimensional objects
    - E.g. a 3-D sphere into a 2-D ellipse: The ellipse has 0 volume (although it does have area)

- Conversely, all matrices of determinant 0 are rank deficient
  – Since they compress full-volumed N-dimensional objects into zero-volume objects

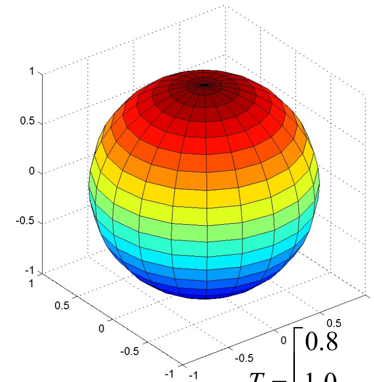
Determinant properties

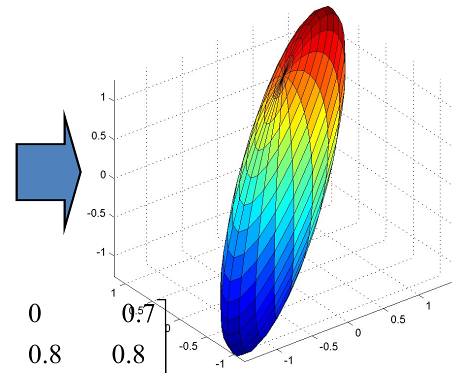
- Associative for square matrices $$|\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}| = |\mathbf{A}|\cdot|\mathbf{B}|\cdot|\mathbf{C}|$$
  - Scaling volume sequentially by several matrices is equal to scaling once by the product of the matrices
- Volume of sum != sum of Volumes $$|(\mathbf{B}+\mathbf{C})| \neq |\mathbf{B}| + |\mathbf{C}|$$
- Commutative
  - The order in which you scale the volume of an object is irrelevant

$$|\mathbf{A}\cdot\mathbf{B}| = |\mathbf{B}\cdot\mathbf{A}| = |\mathbf{A}|\cdot|\mathbf{B}|$$

Matrix Inversion

- A matrix transforms an N-dimensional object to a different N-dimensional object



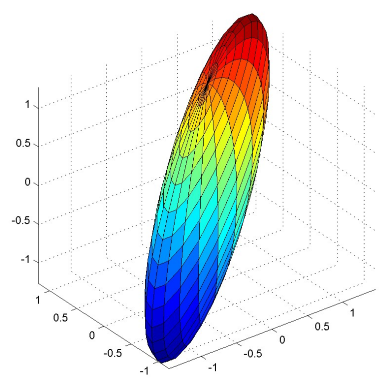


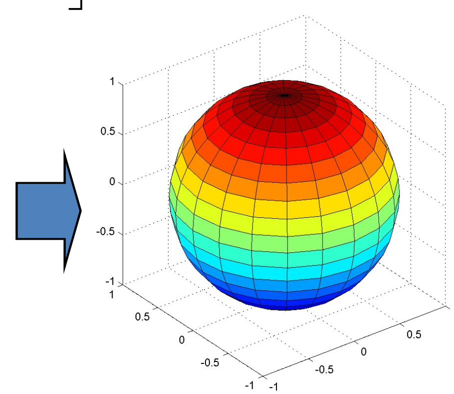
$$T = \begin{bmatrix} 0.8 & 0 & 0.7 \\ 1.0 & 0.8 & 0.8 \\ 0.7 & 0.9 & 0.7 \end{bmatrix}$$

- What transforms the new object back to the original?
  - The *inverse transformation*

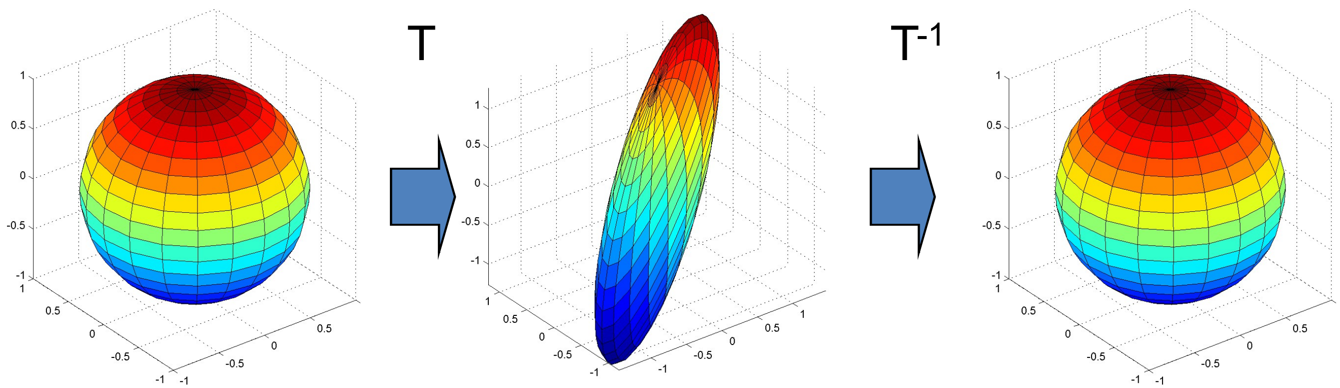
$$Q = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} = T^{-1}$$

- The inverse transformation is called the matrix inverse





Matrix Inversion

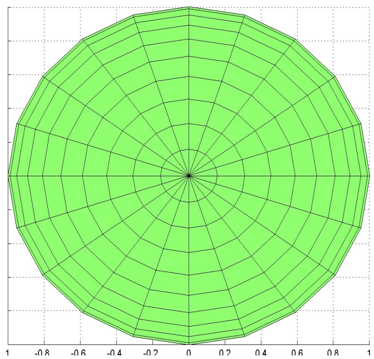


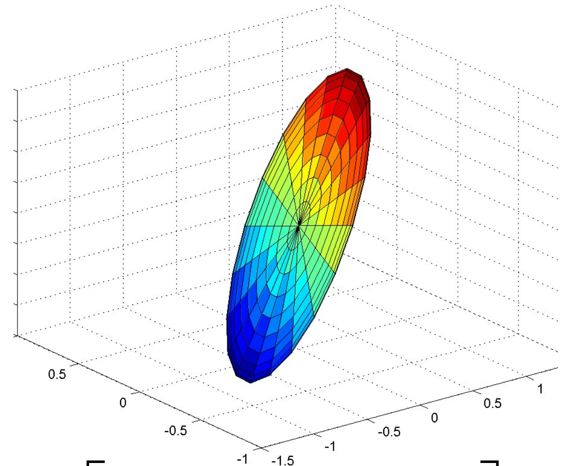
$$\mathbf{T}^{-1}\mathbf{T}\mathbf{D} = \mathbf{D} \Rightarrow \mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$$

- The product of a matrix and its inverse is the identity matrix
  - Transforming an object, and then inverse transforming it gives us back the original object

$$\mathbf{T}\mathbf{T}^{-1}\mathbf{D} = \mathbf{D} \Rightarrow \mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$$

Non-square Matrices





$$\begin{bmatrix} x_1 & x_2 & . & . & x_N \\ y_1 & y_2 & . & . & y_N \end{bmatrix}$$

X = 2D data

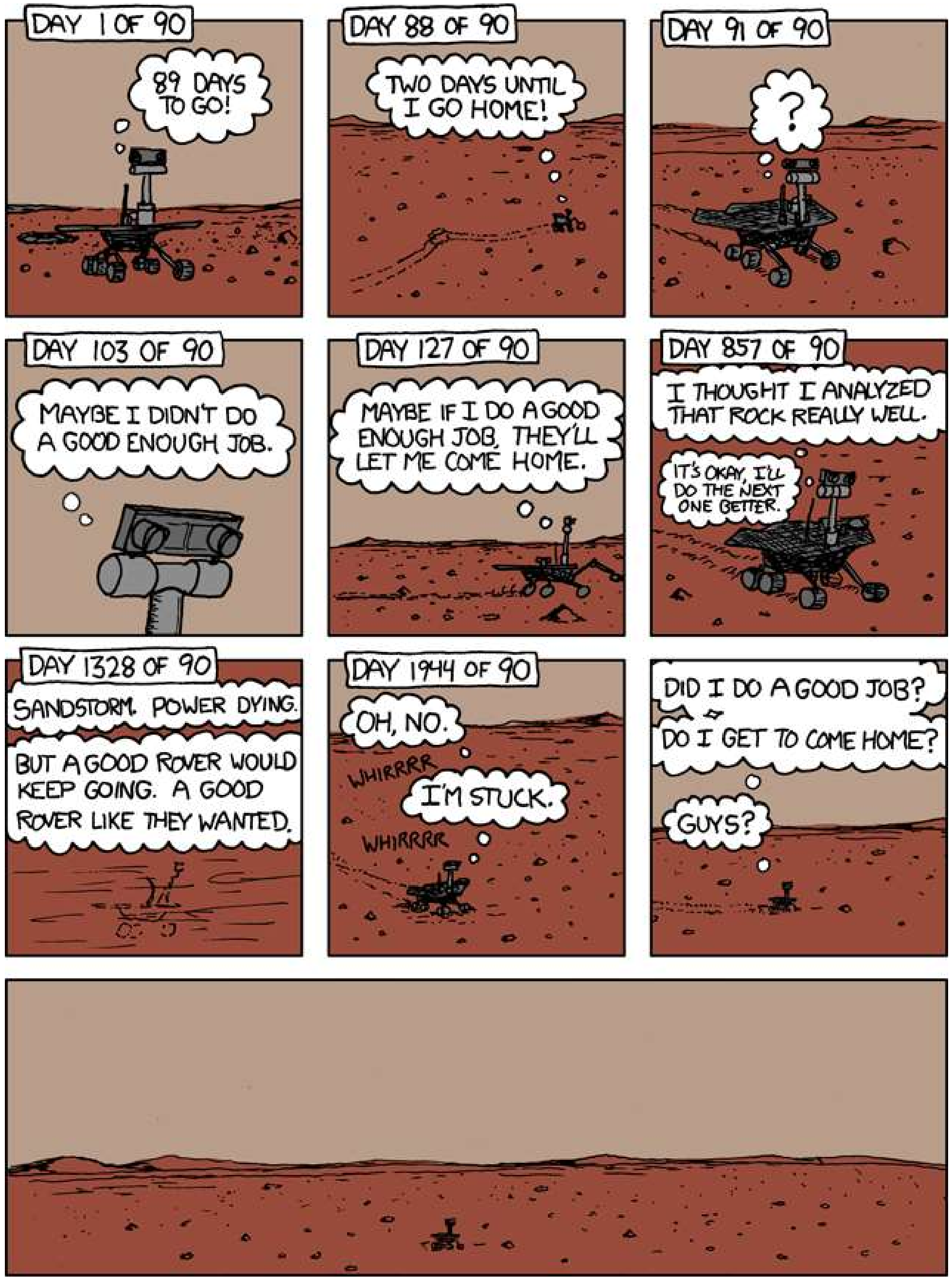
$$\begin{bmatrix} .8 & .9 \\ .1 & .9 \\ .6 & 0 \end{bmatrix}$$

P = transform

$$\begin{bmatrix} \hat{x}_1 & \hat{x}_2 & . & . & \hat{x}_N \\ \hat{y}_1 & \hat{y}_2 & . & . & \hat{y}_N \\ \hat{z}_1 & \hat{z}_2 & . & . & \hat{z}_N \end{bmatrix}$$

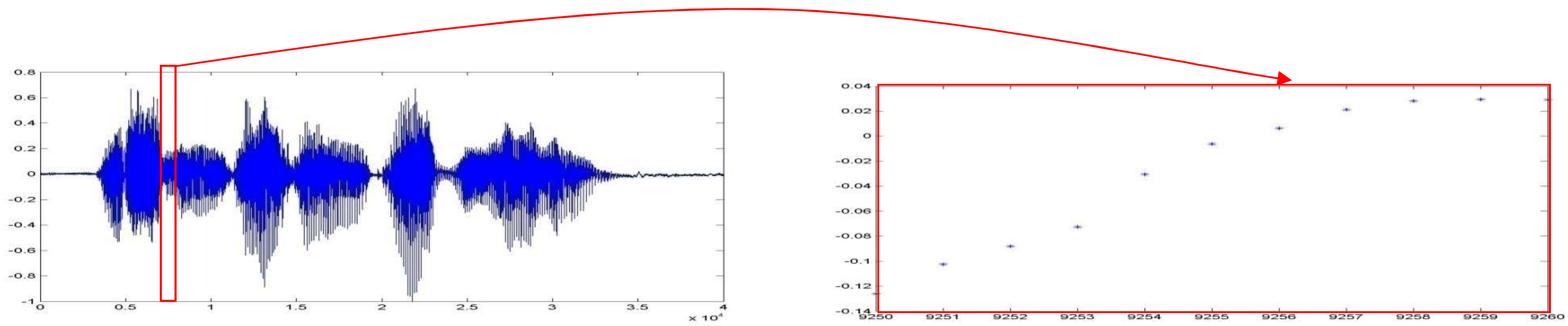
PX = 3D, rank 2

- Non-square matrices add or subtract axes
  - More rows than columns → add axes
    - But does not increase the dimensionality of the data


DAY 1 OF 90
89 DAYS TO GO!
DAY 88 OF 90
TWO DAYS UNTIL I GO HOME!
DAY 91 OF 90
?
DAY 103 OF 90
MAYBE I DIDN'T DO A GOOD ENOUGH JOB.
DAY 127 OF 90
MAYBE IF I DO A GOOD ENOUGH JOB, THEY'LL LET ME COME HOME.
DAY 857 OF 90
I THOUGHT I ANALYZED THAT ROCK REALLY WELL.
IT'S OKAY, I'LL DO THE NEXT ONE BETTER.
DAY 1328 OF 90
SANDSTORM. POWER DYING.
BUT A GOOD ROVER WOULD KEEP GOING. A GOOD ROVER LIKE THEY WANTED.
DAY 1944 OF 90
OH, NO.
WHIRRRR
I'M STUCK.
WHIRRRR
DID I DO A GOOD JOB?
DO I GET TO COME HOME?
GUYS?

Recap: Representing signals as vectors

- Signals are frequently represented as vectors for manipulation
- E.g. A segment of an audio signal

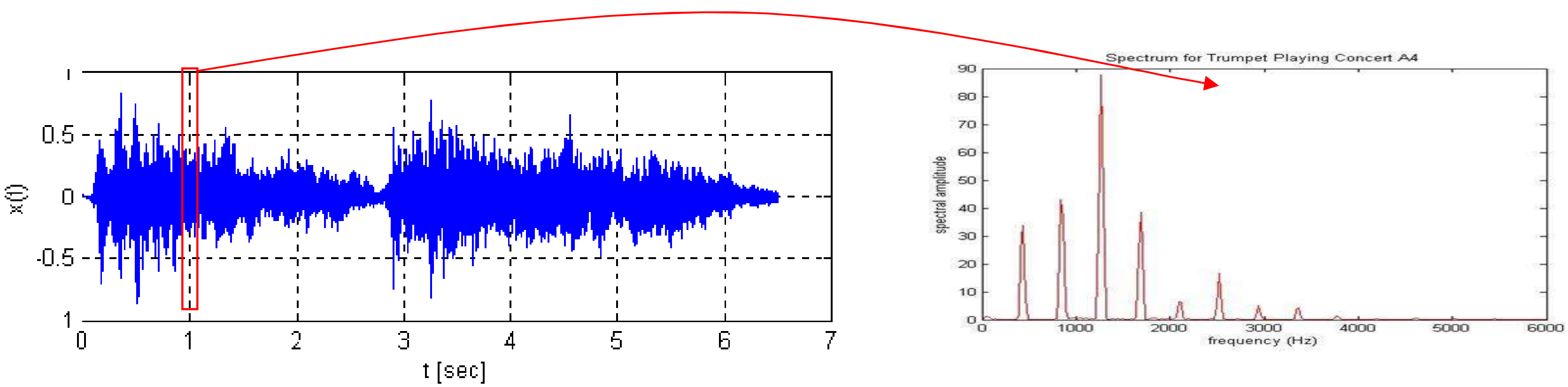


- Represented as a vector of sample values

$$[s_1 \ \ s_2 \ \ s_3 \ \ s_4 \ \dots \ s_N]$$

Representing signals as vectors

- Signals are frequently represented as vectors for manipulation
- E.g. The ***spectrum*** segment of an audio signal



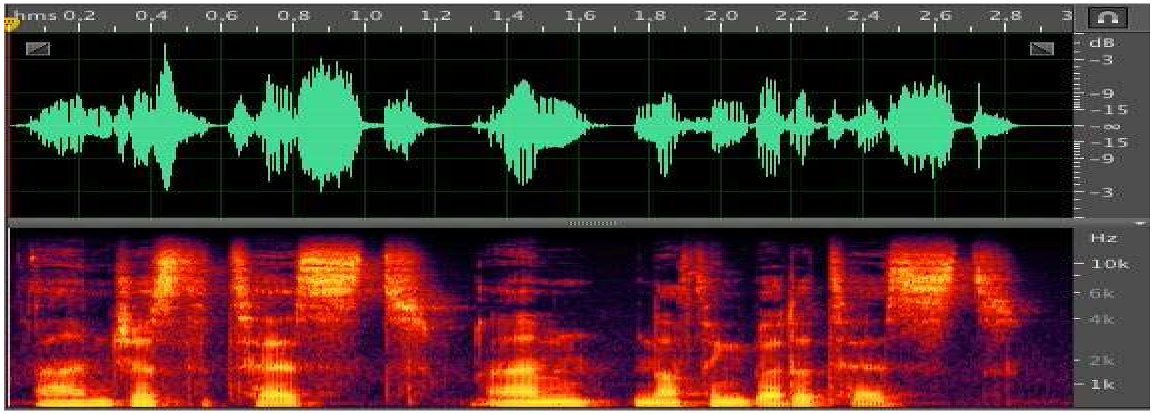
- Represented as a vector of sample values

$$[S_1 \;\; S_2 \;\; S_3 \;\; S_4 \; \ldots \; S_M]$$

  – Each component of the vector represents a frequency component of the spectrum

Representing a signal as a matrix

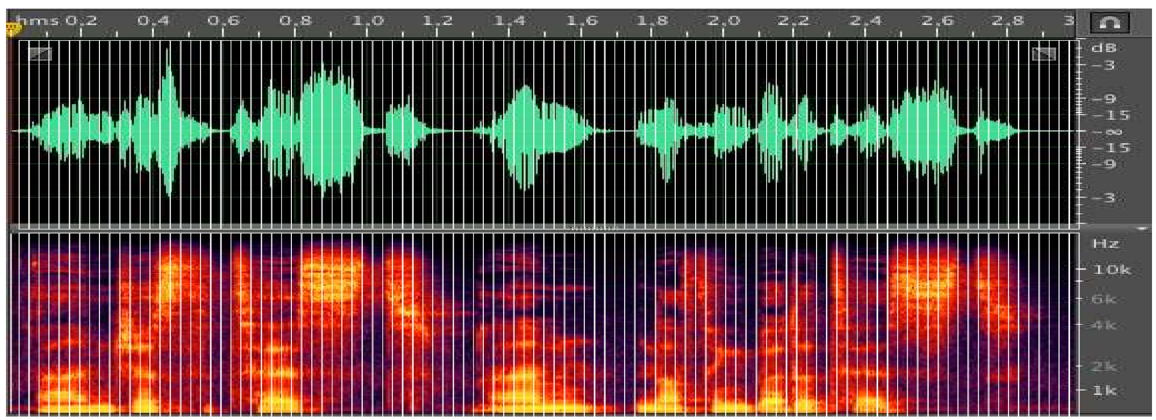
- Time series data like audio signals are often represented as spectrographic matrices



- Each column is the spectrum of a short segment of the audio signal

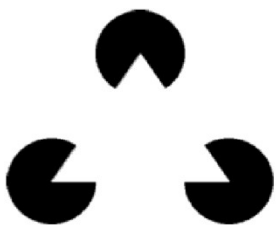
Representing a signal as a matrix

- Time series data like audio signals are often represented as spectrographic matrices



- Each column is the spectrum of a short segment of the audio signal

Representing an image as a vector



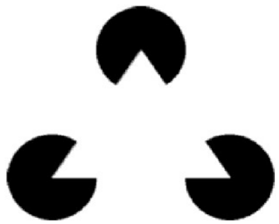
- 3 pacmen
- A 321 x 399 grid of pixel values
  - Row and Column = position
- A 1 x 128079 vector
  - "Unraveling" the matrix

$$\begin{bmatrix}1 & 1 & . & 1 & 1 & . & 0 & 0 & 0 & . & . & 1\end{bmatrix}$$

  - Note: This can be recast as the grid that forms the image

Representing a signal as a matrix

- Images are often just represented as matrices



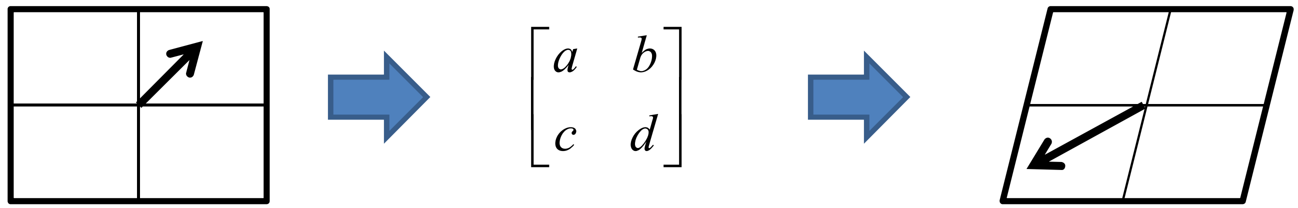
```
>> X(1:32:end,1:40:end)

ans =

     1     1     1     1     1     1     1     1     1     1
     1     1     1     1     0     0     0     1     1     1
     1     1     1     1     0     0     0     1     1     1
     1     1     1     1     0     1     0     1     1     1
     1     1     1     1     1     1     1     1     1     1
     1     1     1     1     1     1     1     1     1     1
     1     1     0     1     1     1     1     1     0     1
     1     0     0     1     1     1     1     1     0     0
     1     0     0     0     1     1     1     0     0     0
     1     0     0     0     1     1     1     0     0     0
     1     1     1     1     1     1     1     1     1     1
```

Interpretations of a matrix

- As a ***transform*** that modifies vectors and vector spaces



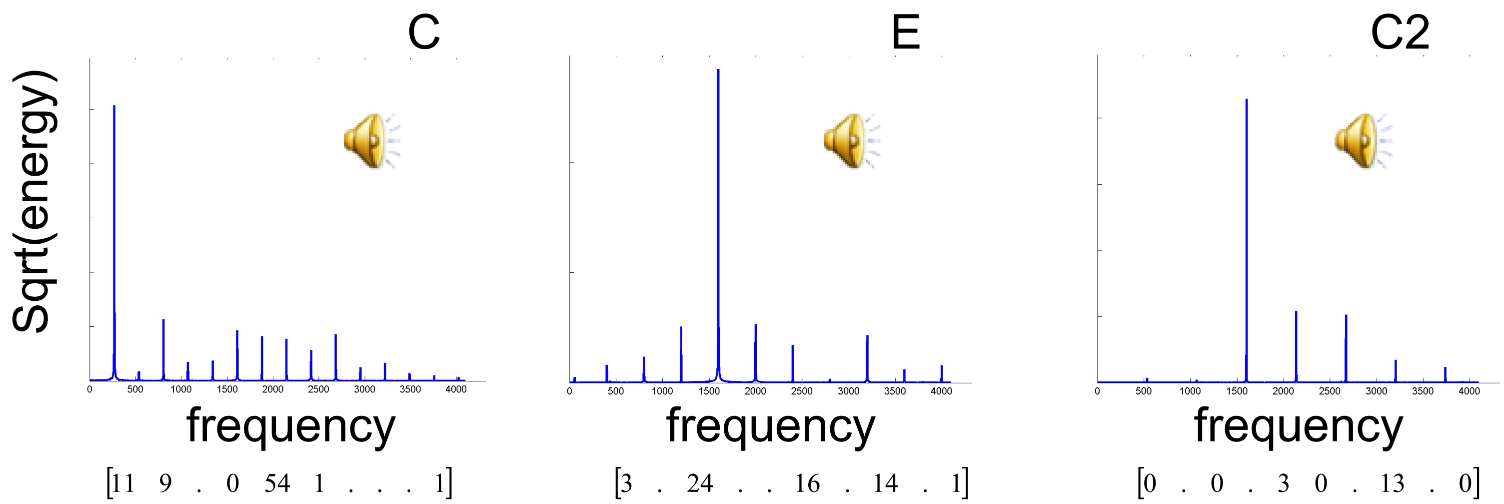
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- As a ***container*** for data (vectors)

$$\begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \end{bmatrix}$$

- As a generator of vector spaces..

Revise.. Vector dot product

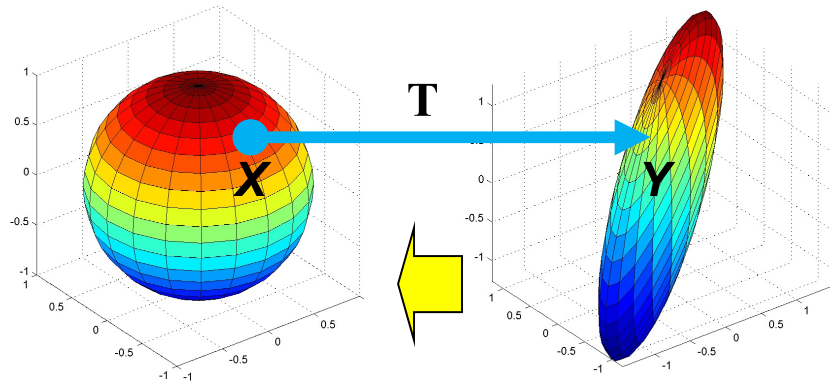


- How much of C is also in E
  - How much can you fake a C by playing an E
  - C.E / |C||E| = 0.1
  - Not very much
- How much of C is in C2?
  - C.C2 / |C| /|C2| = 0.5
  - Not bad, you can fake it

Overview

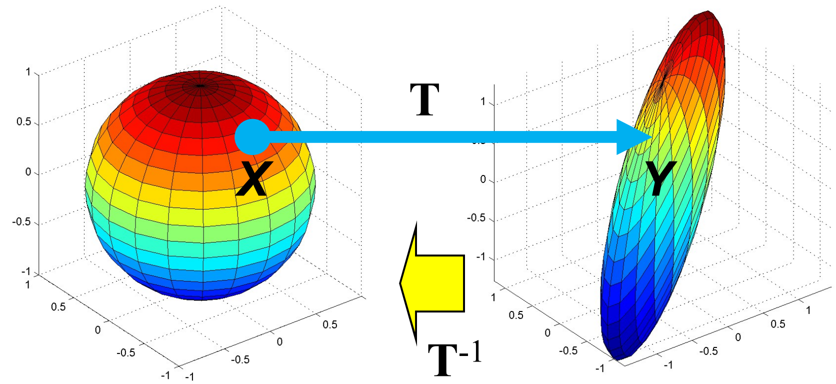
- Vectors and matrices
- Basic vector/matrix operations
- Various matrix types
- Matrix properties
  - Determinant
  - Inverse
  - Rank
- **Solving simultaneous equations**
- Projections
- Eigen decomposition
- SVD

The Inverse Transform and Simultaneous Equations



- Given the Transform $\mathbf{T}$ and transformed vector ***Y***, how do we determine ***X***?

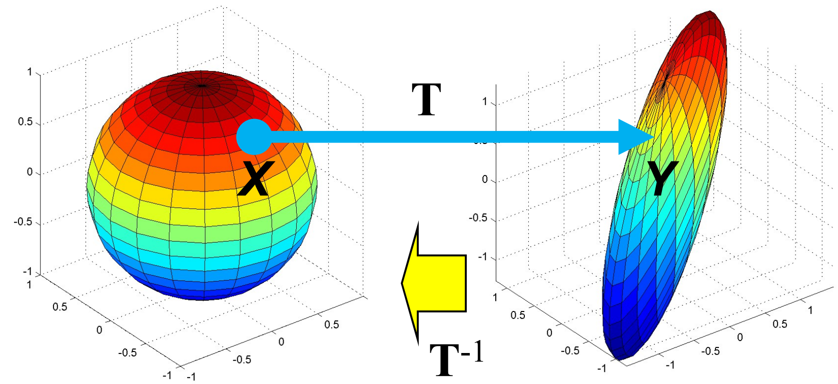
Matrix inversion (division)



- The inverse of matrix multiplication
  - Not element-wise division!!
  - E.g.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$$

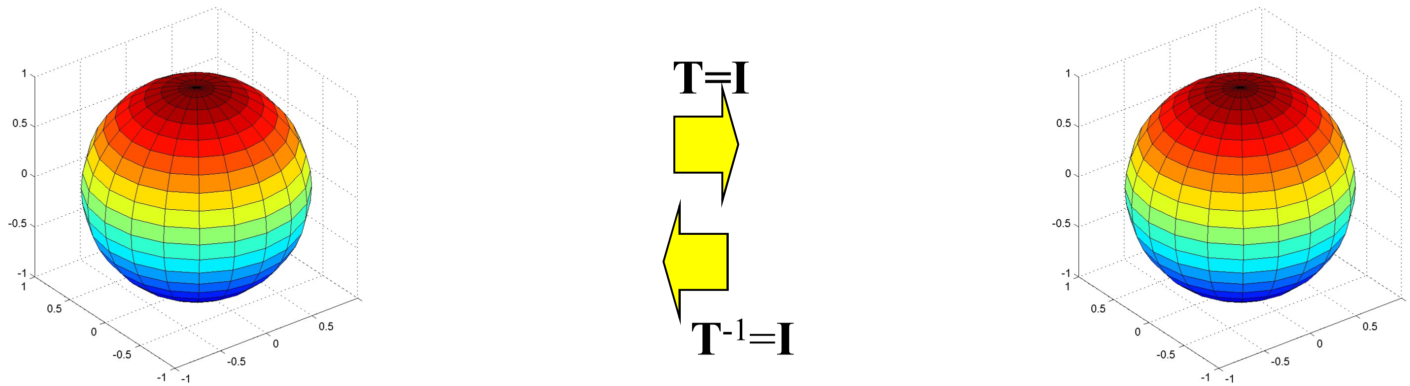
Matrix inversion (division)



- Provides a way to "undo" a linear transform
- Undoing a transform must happen as soon as it is performed
- Effect on matrix inversion: Note order of multiplication

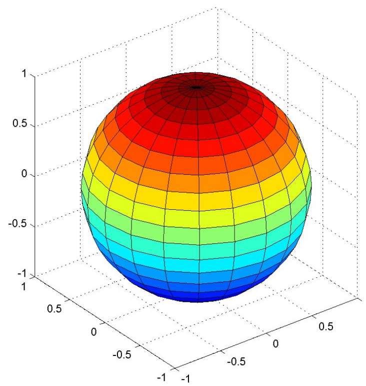
$$\mathbf{A}\cdot\mathbf{B}=\mathbf{C},\ \ \mathbf{A}=\mathbf{C}\cdot\mathbf{B}^{-1},\ \ \mathbf{B}=\mathbf{A}^{-1}\cdot\mathbf{C}$$

Matrix inversion (division)

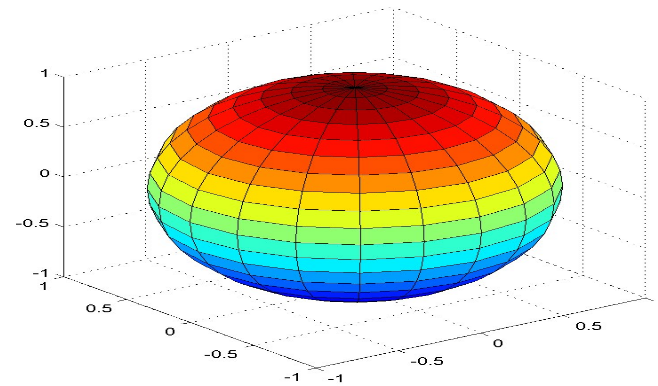


- Inverse of the unit matrix is itself

Matrix inversion (division)



$$\mathbf{T} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

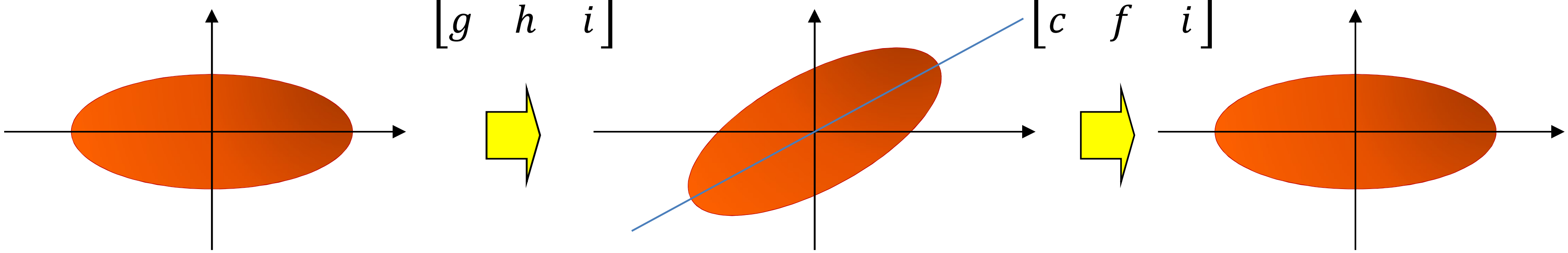


$$\mathbf{T}^{-1} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Inverse of the unit matrix is itself
- Inverse of a diagonal is diagonal

Matrix inversion (division)

$$\mathbf{R} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \mathbf{R}^{-1} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

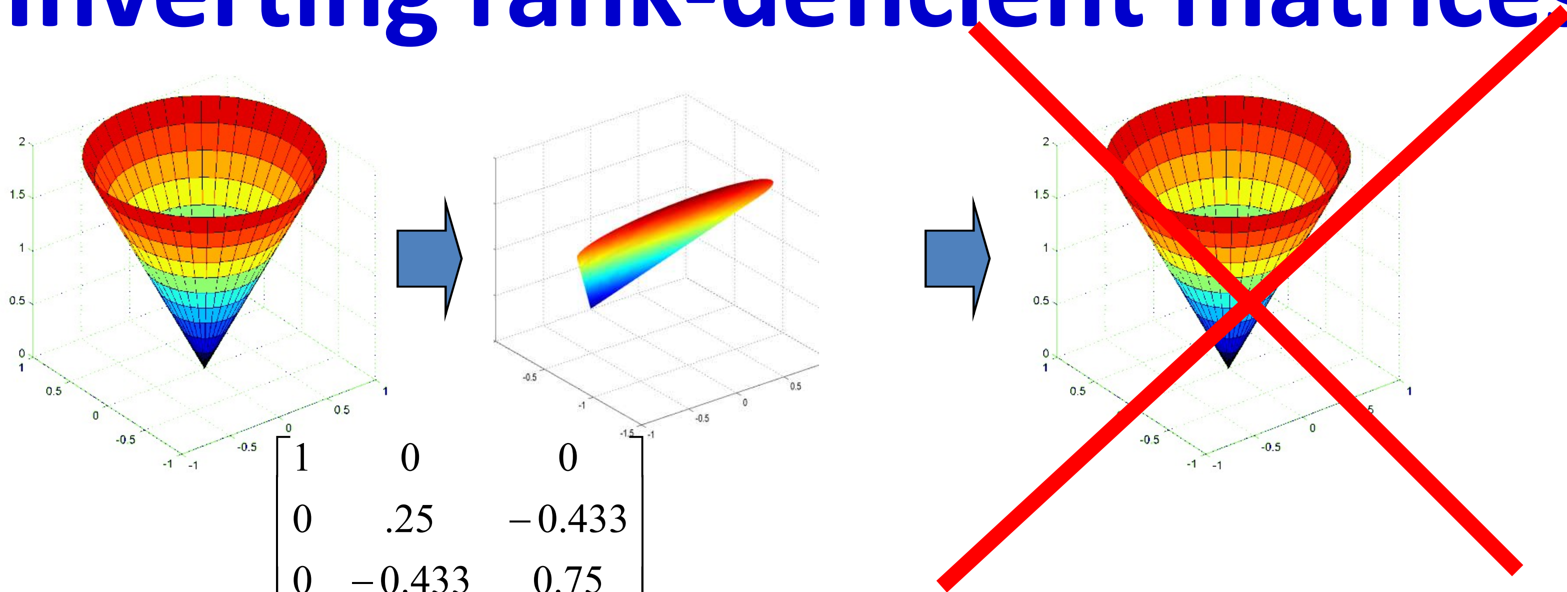


- Inverse of the unit matrix is itself
- Inverse of a diagonal is diagonal
- Inverse of a rotation is a (counter)rotation (its transpose!)
  - In 2D a forward rotation θ by is cancelled by a backward rotation of $-\theta$

$$\mathbf{R} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}, \mathbf{R}^{-1} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

  - More generally, in any number of dimensions: $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$

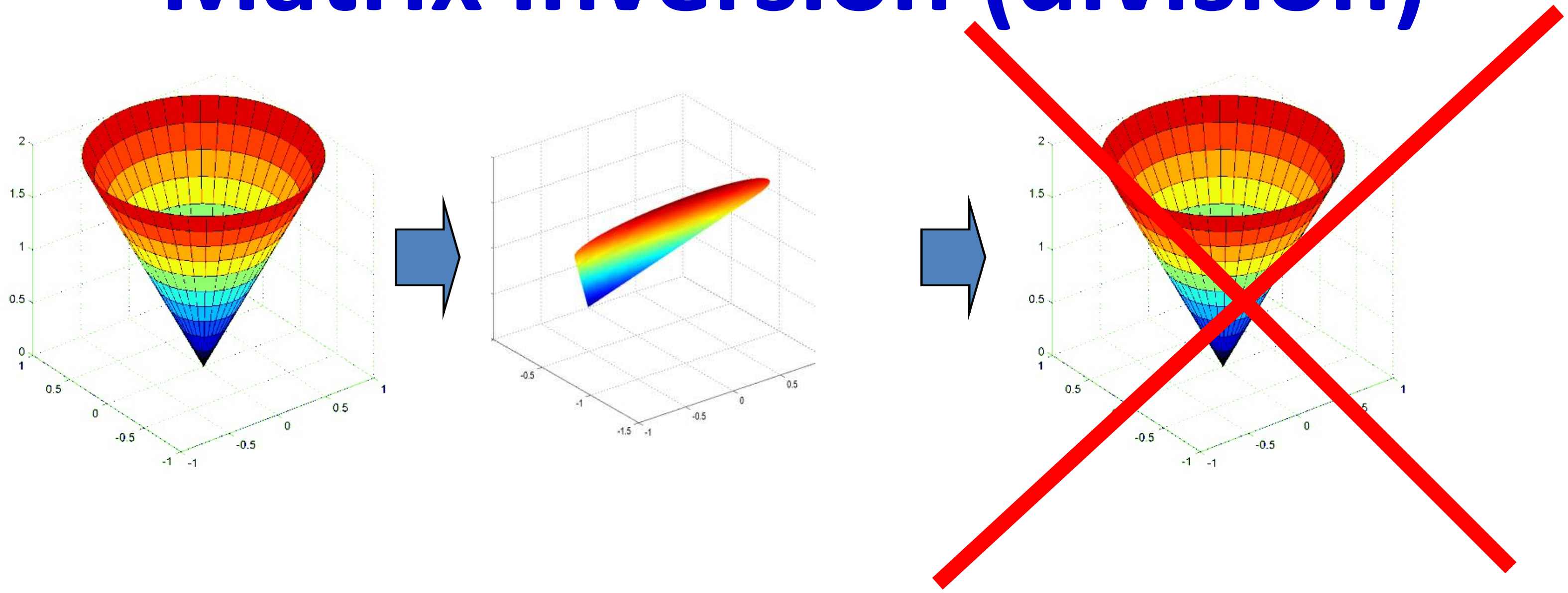
Inverting rank-deficient matrices



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & .25 & -0.433 \\ 0 & -0.433 & 0.75 \end{bmatrix}$$

- Rank deficient matrices “flatten” objects
  - In the process, multiple points in the original object get mapped to the same point in the transformed object
- It is not possible to go “back” from the flattened object to the original object
  - Because of the many-to-one forward mapping
- Rank deficient matrices have no inverse

Matrix inversion (division)



- Inverse of the unit matrix is itself
- Inverse of a diagonal is diagonal
- Inverse of a rotation is a (counter)rotation (its transpose!)
- Inverse of a rank deficient matrix does not exist!

Inverse Transform and Simultaneous Equation

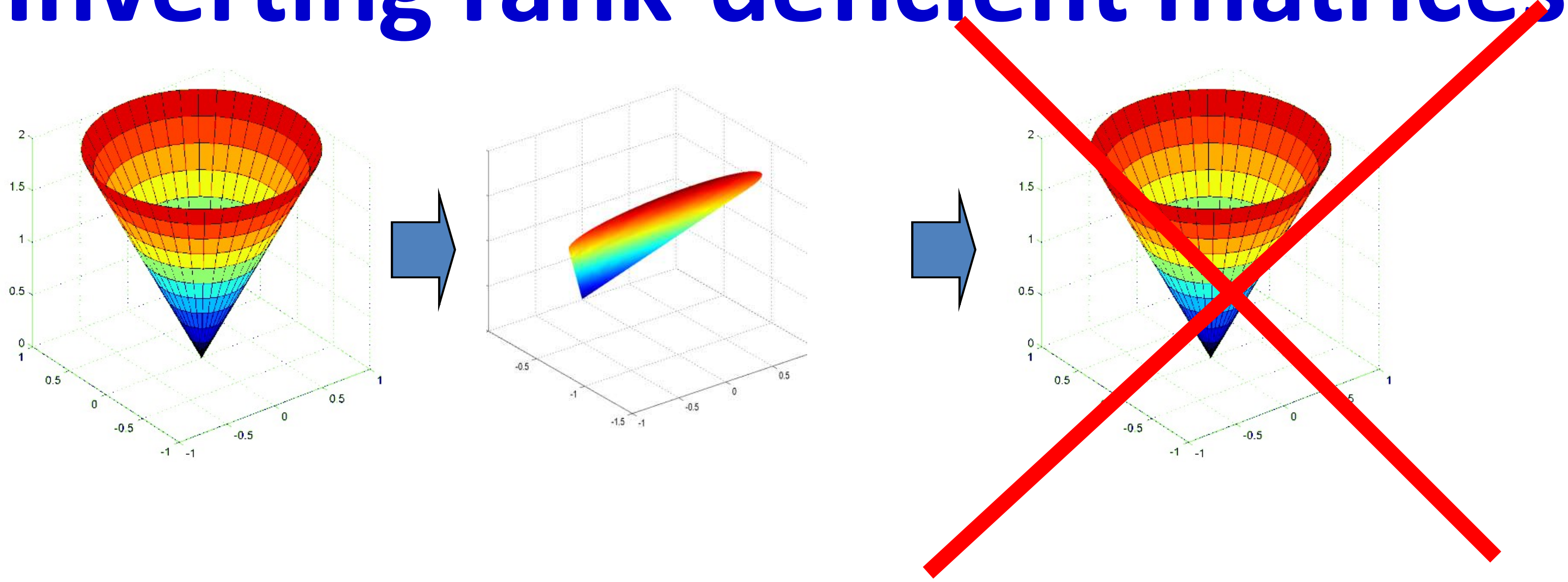
$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{array}{l} a = T_{11}x + T_{12}y + T_{13}z \\ b = T_{21}x + T_{22}y + T_{23}z \\ c = T_{31}x + T_{32}y + T_{33}z \end{array}$$

Given $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\mathbf{T}$ find $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- Inverting the transform is identical to solving simultaneous equations

Inverting rank-deficient matrices



- Rank deficient matrices have no inverse
  - In this example, there is no *unique* inverse

Inverse Transform and Simultaneous Equation

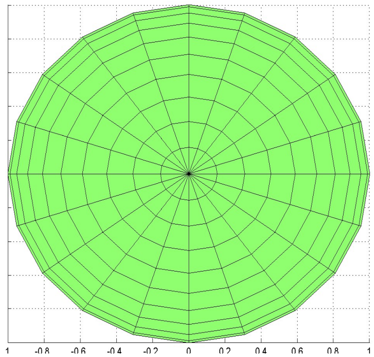
$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \end{bmatrix}$$

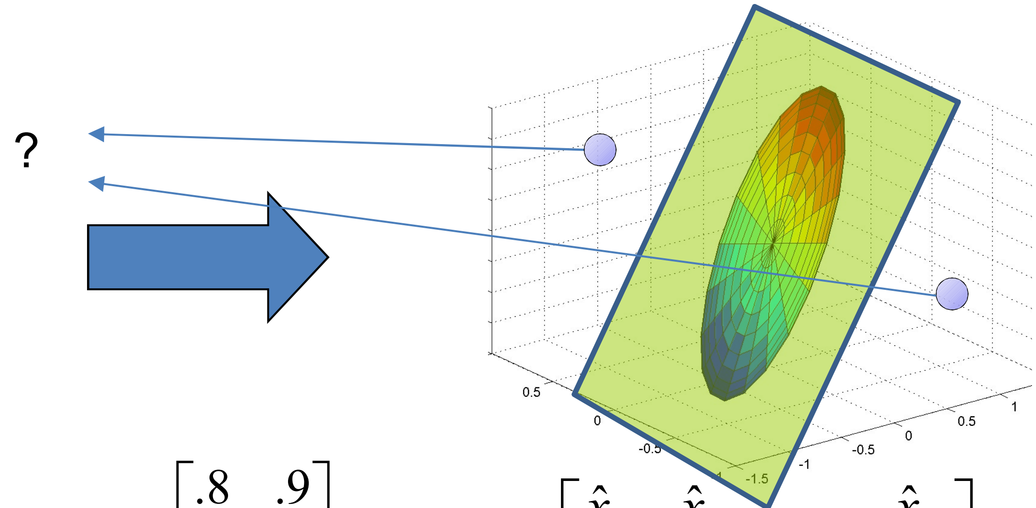
$$\begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{matrix} a = T_{11}x + T_{12}y + T_{13}z \\ b = T_{21}x + T_{22}y + T_{23}z \end{matrix}$$

Given $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{T}$ find $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- Inverting the transform is identical to solving simultaneous equations
- Rank-deficient transforms result in too-few ***independent*** equations
  - Cannot be inverted to obtain a *unique* solution

Non-square Matrices





$$\begin{bmatrix} x_1 & x_2 & . & . & x_N \\ y_1 & y_2 & . & . & y_N \end{bmatrix}$$

X = 2D data

$$\begin{bmatrix} .8 & .9 \\ .1 & .9 \\ .6 & 0 \end{bmatrix}$$

P = transform

$$\begin{bmatrix} \hat{x}_1 & \hat{x}_2 & . & . & \hat{x}_N \\ \hat{y}_1 & \hat{y}_2 & . & . & \hat{y}_N \\ \hat{z}_1 & \hat{z}_2 & . & . & \hat{z}_N \end{bmatrix}$$

PX = 3D, rank 2

- When the transform *increases* the number of components most points in the new space will not have a corresponding preimage

Inverse Transform and Simultaneous Equation

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \\ T_{31} & T_{32} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{matrix} a = T_{11}x + T_{12}y \\ b = T_{21}x + T_{22}y \\ c = T_{31}x + T_{32}y \end{matrix}$$

Given $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\mathbf{T}$ find $\begin{bmatrix} x \\ y \end{bmatrix}$

- Inverting the transform is identical to solving simultaneous equations
- Rank-deficient transforms result in too few independent equations
  - Cannot be inverted to obtain a unique solution
- Or too *many* equations
  - Cannot be inverted to obtain an exact solution

The Pseudo Inverse (PINV)

$$V \approx T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} \approx Pinv(T)V$$

- When you can't *really* invert T, you perform the *pseudo* inverse

Generalization to matrices

- **Unique exact solution exists**
- **T** must be square

$$\mathbf{X} = \mathbf{TY} \Rightarrow \mathbf{Y} = \mathbf{T}^{-1}\mathbf{X}$$

Left multiplication

$$\mathbf{X} = \mathbf{YT} \Rightarrow \mathbf{Y} = \mathbf{XT}^{-1}$$

Right multiplication

- **No unique exact solution exists**
  - At least one (if not both) of the forward and backward equations may be inexact
- **T** may or may not be square

$$\mathbf{X} = \mathbf{TY} \Rightarrow \mathbf{Y} = \mathrm{Pinv}(\mathbf{T})\mathbf{X}$$

Left multiplication

$$\mathbf{X} = \mathbf{YT} \Rightarrow \mathbf{Y} = \mathbf{X}\mathrm{Pinv}(\mathbf{T})$$

Right multiplication

Underdetermined Pseudo Inverse

$$\begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{aligned} a &= T_{11}x + T_{12}y + T_{13}z \\ b &= T_{21}x + T_{22}y + T_{23}z \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = Pinv(\mathbf{T}) \begin{bmatrix} a \\ b \end{bmatrix}$$

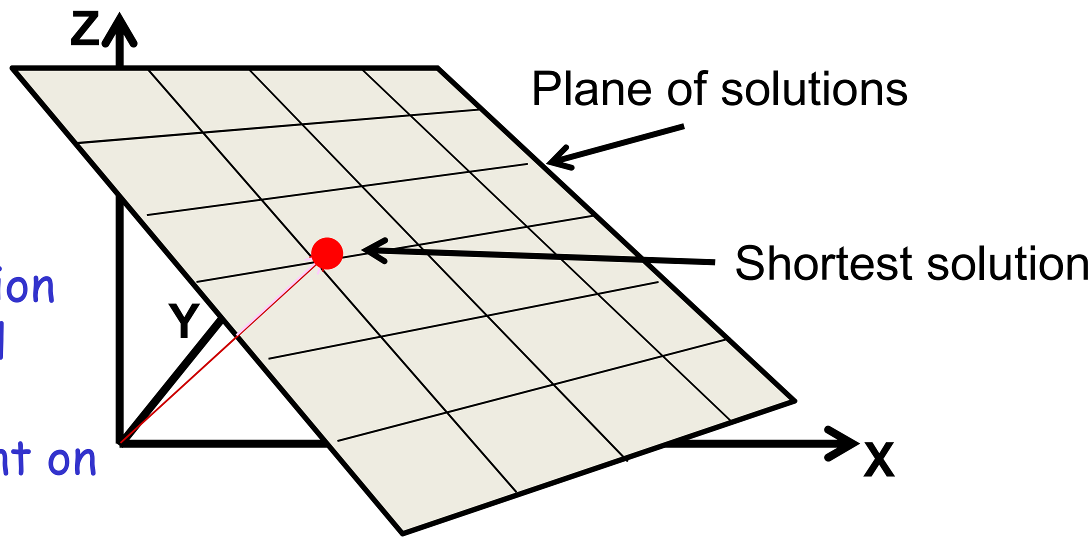


Figure only meant for illustration for the above equations, actual set of solutions is a line, not a plane. Pinv(T)A will be the point on the line closest to origin

- **Case 1:** Too many solutions
- Pinv(**T**)**A** picks the *shortest* solution

The Pseudo Inverse for the underdetermined case

$$\begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{matrix} a = T_{11}x + T_{12}y + T_{13}z \\ b = T_{21}x + T_{22}y + T_{23}z \end{matrix}$$

$$V \approx T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = Pinv(T)V$$

$$Pinv(\boldsymbol{T}) = \boldsymbol{T}^T (\boldsymbol{T}\boldsymbol{T}^T)^{-1}$$

$$\boldsymbol{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{T} Pinv(\boldsymbol{T}) \boldsymbol{V} = \boldsymbol{T}\boldsymbol{T}^T (\boldsymbol{T}\boldsymbol{T}^T)^{-1} \boldsymbol{V} = \boldsymbol{V}$$

The Pseudo Inverse

$$\mathbf{T}=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \\ T_{31} & T_{32} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{T}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{array}{l} a = T_{11}x + T_{12}y \\ b = T_{21}x + T_{22}y \\ c = T_{31}x + T_{32}y \end{array}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = Pinv(\mathbf{T})\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$||\mathbf{A} - \mathbf{TX}||^2$

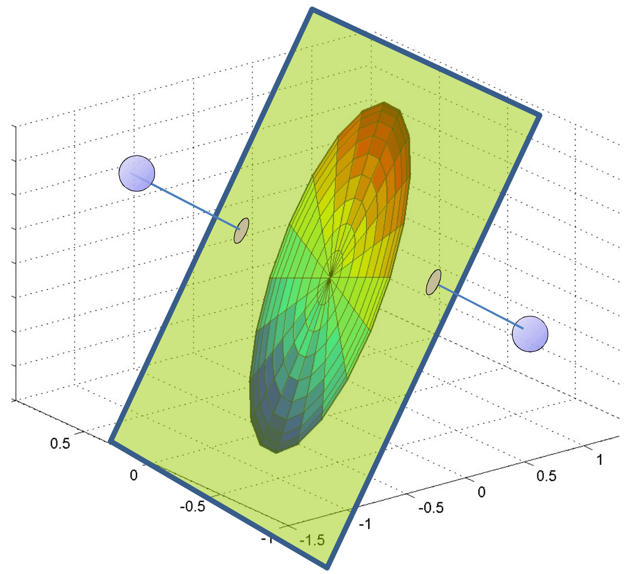


Figure only meant for illustration for the above equations, Pinv(T) will actually have 6 components. The error is a quadratic in 6 dimensions

- **Case 2:** No exact solution
- $\mathrm{Pinv}(\mathbf{T})\mathbf{A}$ picks the solution that results in the lowest error

The Pseudo Inverse for the overdetermined case

$$E = \|\boldsymbol{TX} - \boldsymbol{A}\|^2 = (\boldsymbol{TX} - \boldsymbol{A})^T(\boldsymbol{TX} - \boldsymbol{A})$$

$$E = \boldsymbol{X}^T\boldsymbol{T}^T\boldsymbol{TX} - 2\boldsymbol{X}^T\boldsymbol{T}^T\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{A}$$

Differentiating and equating to 0 we get:

$$\boldsymbol{X} = (\boldsymbol{T}^T\boldsymbol{T})^{-1}\boldsymbol{T}^T\boldsymbol{A} \quad = \quad Pinv(\boldsymbol{T})\boldsymbol{A}$$

$$Pinv(\boldsymbol{T}) = (\boldsymbol{T}^T\boldsymbol{T})^{-1}\boldsymbol{T}^T$$

Shortcut: overdetermined case

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{T}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{matrix} a = T_{11}x + T_{12}y \\ b = T_{21}x + T_{22}y \\ c = T_{31}x + T_{32}y \end{matrix}$$

$$\boldsymbol{V} \approx \boldsymbol{T}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \boldsymbol{T}^T\boldsymbol{V} \approx \boldsymbol{T}^T\boldsymbol{T}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = (\boldsymbol{T}^T\boldsymbol{T})^{-1}\boldsymbol{T}^T\boldsymbol{V}$$

$$Pinv(\boldsymbol{T}) = (\boldsymbol{T}^T\boldsymbol{T})^{-1}\boldsymbol{T}^T$$

Note that in this case:

$$\boldsymbol{T}\begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{T}Pinv(\boldsymbol{T})\boldsymbol{V} = \boldsymbol{T}(\boldsymbol{T}^T\boldsymbol{T})^{-1}\boldsymbol{T}^T\boldsymbol{V} \neq \boldsymbol{V}$$

Why?

Overdetermined vs Underdetermined

- Underdetermined case: Exact solution exists. We find *one* of the exact solutions. Hence..

$$\boldsymbol{T}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{T}Pinv(\boldsymbol{T})\boldsymbol{V} = \boldsymbol{T}\boldsymbol{T}^T(\boldsymbol{T}\boldsymbol{T}^T)^{-1}\boldsymbol{V} = \boldsymbol{V}$$

- Overdetermined case: Solution generally does not exist. Solution is only an approximation..

$$\boldsymbol{T}\begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{T}Pinv(\boldsymbol{T})\boldsymbol{V} = \boldsymbol{T}(\boldsymbol{T}^T\boldsymbol{T})^{-1}\boldsymbol{T}^T\boldsymbol{V} \neq \boldsymbol{V}$$

Properties of the Pseudoinverse

- For the underdetermined case:

$$\boldsymbol{T}Pinv(\boldsymbol{T}) = \mathbf{I}$$

- For the overdetermined case

$$\boldsymbol{T}Pinv(\boldsymbol{T}) = ?$$

  - We return to this question shortly

Matrix inversion (division)

- The inverse of matrix multiplication
  - Not element-wise division!!
- Provides a way to "undo" a linear transformation

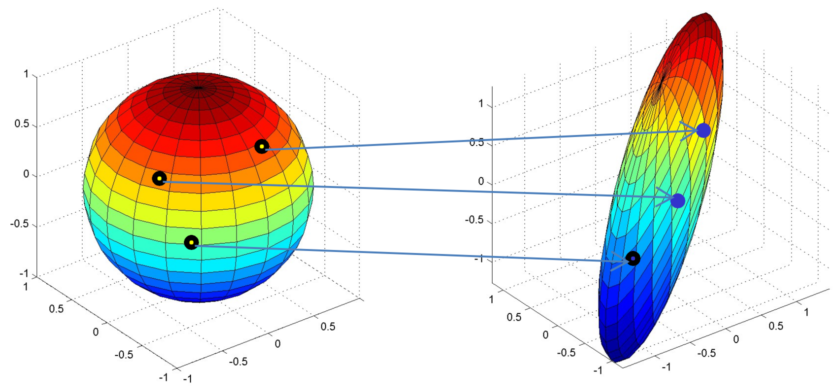
- For square matrices: Pay attention to multiplication side!

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{C}, \;\; \mathbf{A} = \mathbf{C} \cdot \mathbf{B}^{-1}, \;\; \mathbf{B} = \mathbf{A}^{-1} \cdot \mathbf{C}$$

- If matrix is not square use a matrix pseudoinverse:

$$\mathbf{A} \cdot \mathbf{B} \approx \mathbf{C}, \; \mathbf{A} = \mathbf{C} \cdot \mathbf{B}^{+}, \; \mathbf{B} = \mathbf{A}^{+} \cdot \mathbf{C}$$

Finding the Transform

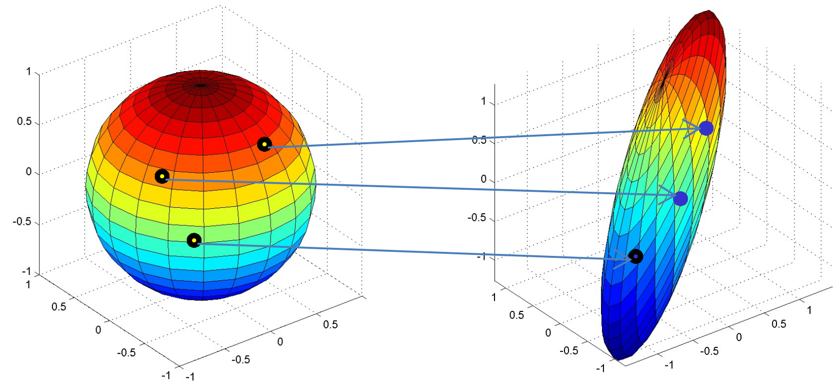


- Given examples
  - $\mathbf{T}.X_1 = Y_1$
  - $\mathbf{T}.X_2 = Y_2$
  - ..
  - $\mathbf{T}.X_N = Y_N$
- Find **T**

Finding the Transform

$$\mathbf{X} = \begin{bmatrix} \uparrow & \vdots & \uparrow \\ X_1 & \ddots & X_N \\ \downarrow & \vdots & \downarrow \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} \uparrow & \vdots & \uparrow \\ Y_1 & \ddots & Y_N \\ \downarrow & \vdots & \downarrow \end{bmatrix}$$



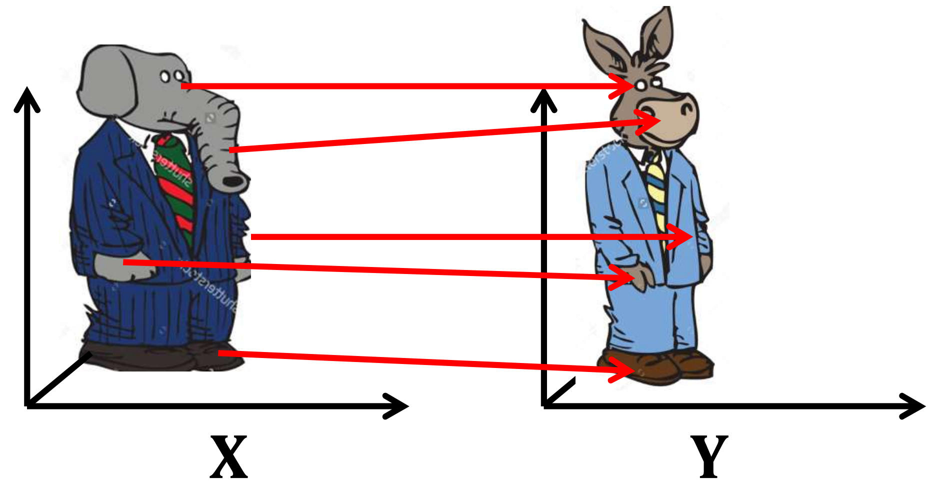
$$\mathbf{Y} = \mathbf{TX} \qquad \mathbf{T} = \mathbf{Y}\mathrm{Pinv}(\mathbf{X})$$

- Pinv works here too

Finding the Transform: Inexact

$$\mathbf{X} = \begin{bmatrix} \uparrow & \vdots & \uparrow \\ X_1 & \ddots & X_N \\ \downarrow & \vdots & \downarrow \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} \uparrow & \vdots & \uparrow \\ Y_1 & \ddots & Y_N \\ \downarrow & \vdots & \downarrow \end{bmatrix}$$



$$\mathbf{Y} \approx \mathbf{TX} \Rightarrow \mathbf{T} = \mathbf{Y}\mathrm{Pinv}(\mathbf{X})$$

$$\text{minimizes } \sum_i ||\mathrm{Y_i} - \mathbf{T}\mathrm{X_i}||^2$$

- Even works for inexact solutions
- We *desire* to find a linear transform **T** that maps **X** to **Y**
  - But such a linear transform doesn't really exist
- *Pinv* will give us the "best guess" for **T** that minimizes the total squared error between **Y** and **TX**

Overview

- Vectors and matrices
- Basic vector/matrix operations
- Various matrix types
- Matrix properties
  - Determinant
  - Inverse
  - Rank
- Solving simultaneous equations
- **Projections**
- Eigen decomposition
- SVD

Flashback: The *true* representation of a vector

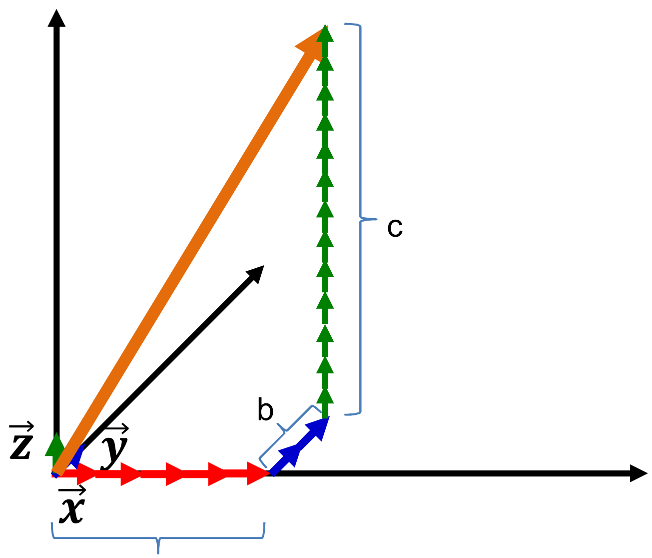
$$\boldsymbol{v} = d\vec{\boldsymbol{u}} + e\vec{\boldsymbol{v}} + f\vec{\boldsymbol{w}}$$

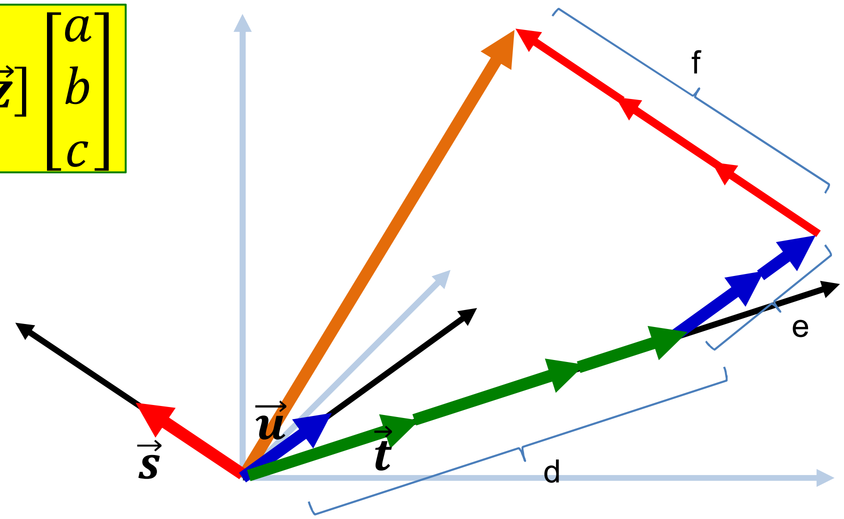
$$\boldsymbol{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{using } \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}, \vec{\boldsymbol{z}}$$

$$\boldsymbol{v} = a\vec{\boldsymbol{x}} + b\vec{\boldsymbol{y}} + c\vec{\boldsymbol{z}}$$

$$\boldsymbol{v} = [\vec{\boldsymbol{x}} \quad \vec{\boldsymbol{y}} \quad \vec{\boldsymbol{z}}] \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$





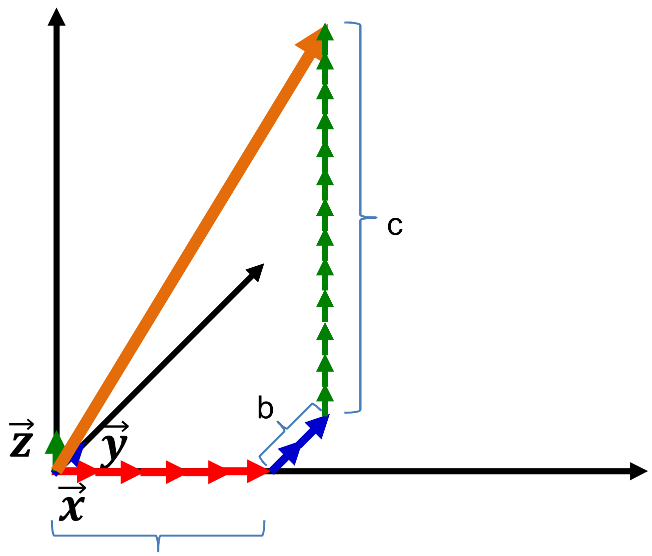
$$\boldsymbol{v} = [\vec{\boldsymbol{s}} \quad \vec{\boldsymbol{t}} \quad \vec{\boldsymbol{u}}] \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

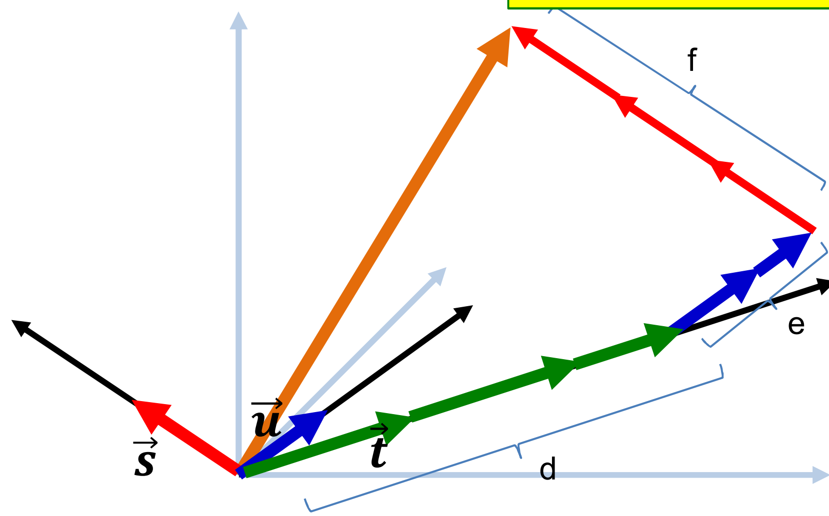
- What the column (or row) of numbers really means
  - The “basis matrix” is implicit

Flashforward: Changing bases

$$\boldsymbol{v} = [\vec{\boldsymbol{x}} \quad \vec{\boldsymbol{y}} \quad \vec{\boldsymbol{z}}] \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\boldsymbol{v} = [\vec{\boldsymbol{s}} \quad \vec{\boldsymbol{t}} \quad \vec{\boldsymbol{u}}] \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$





- Given representation $[a, b, c]$ and bases $\vec{\boldsymbol{x}} \quad \vec{\boldsymbol{y}} \quad \vec{\boldsymbol{z}}$, how do we derive the representation $[d \; e \; f]$ in terms of a different set of bases $\vec{\boldsymbol{s}} \quad \vec{\boldsymbol{t}} \quad \vec{\boldsymbol{u}}$?

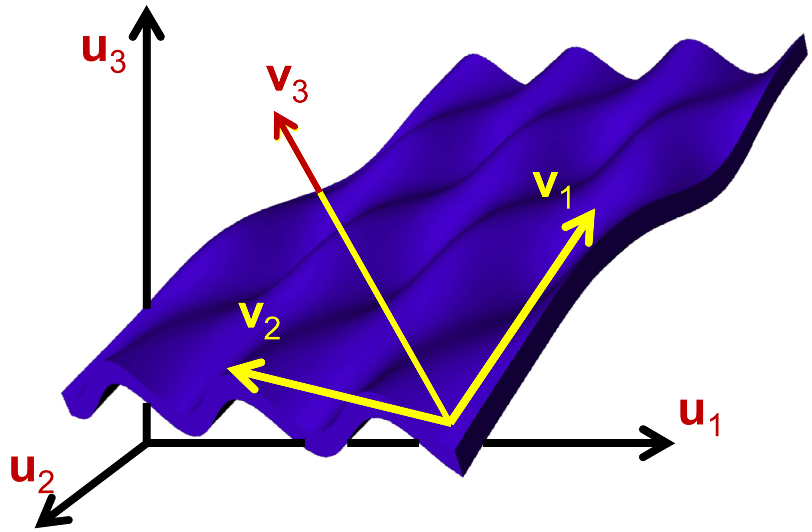
Matrix as a Basis transform

$$\mathbf{X} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3, \quad \Longleftarrow \quad \mathbf{X} = x\mathbf{u}_1 + y\mathbf{u}_2 + z\mathbf{u}_3$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

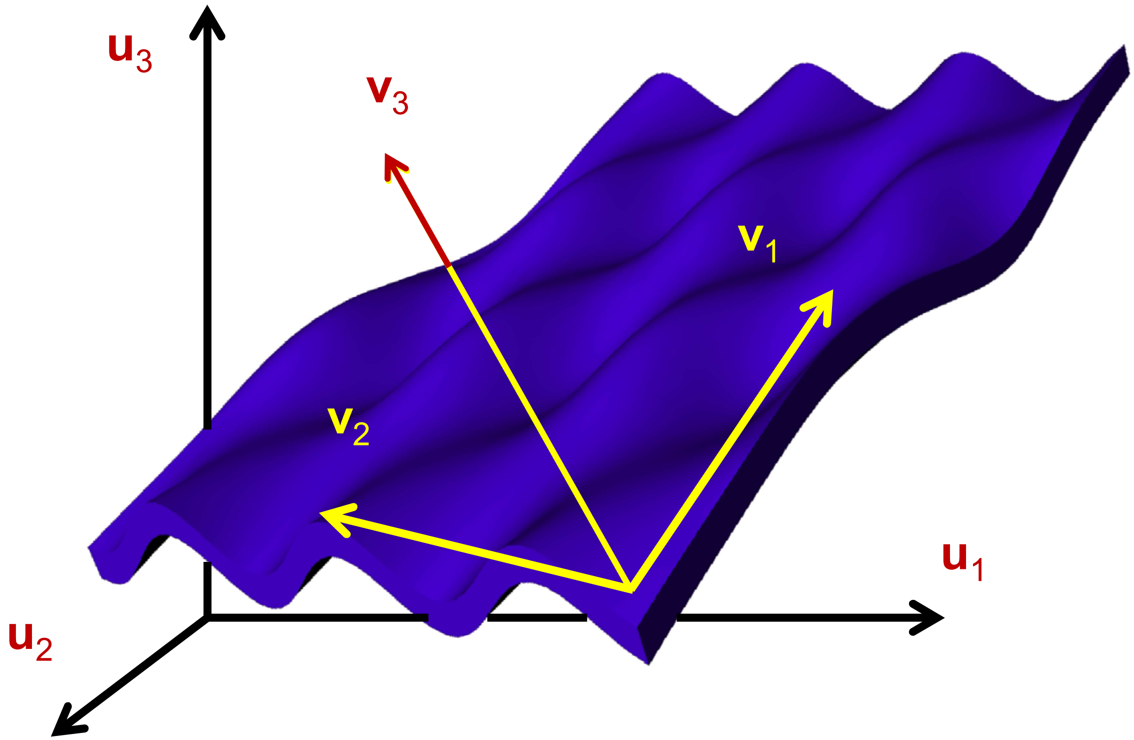
- A matrix transforms a representation in terms of a standard basis $\mathbf{u}_1$ $\mathbf{u}_2$ $\mathbf{u}_3$ to a representation in terms of a different bases $\mathbf{v}_1$ $\mathbf{v}_2$ $\mathbf{v}_3$
- Finding best bases: Find matrix that transforms standard representation to these bases

Basis based representation



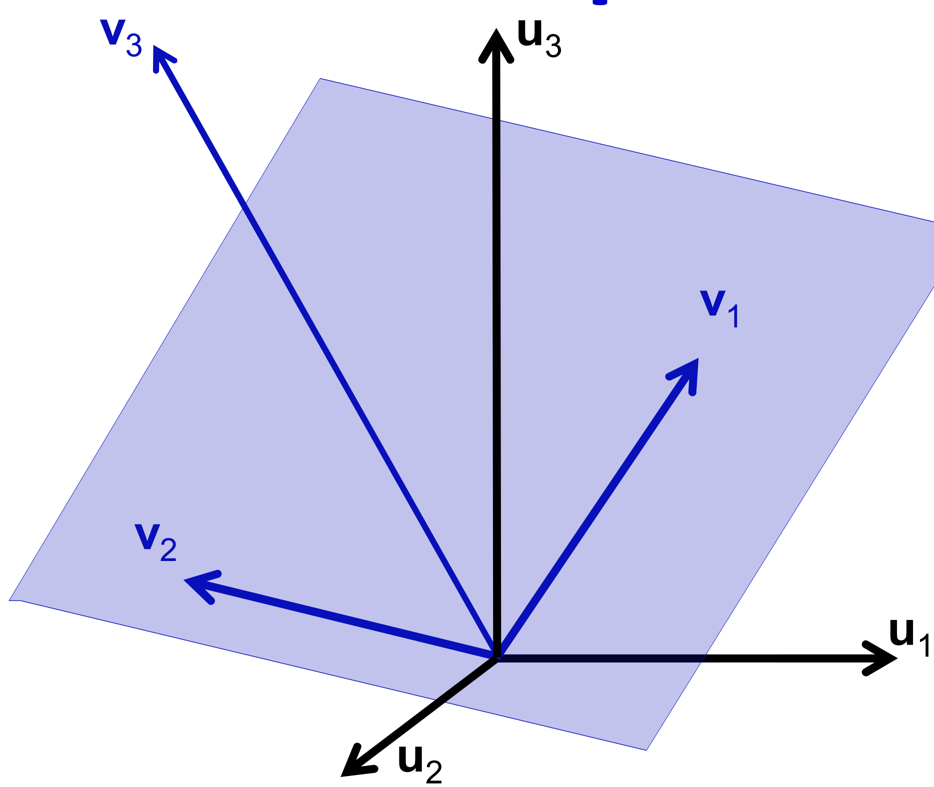
- A "good" basis captures *data* structure
- Here $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ all take large values for data in the set
- But in the ($\mathbf{v}_1$ $\mathbf{v}_2$ $\mathbf{v}_3$) set, coordinate values along $\mathbf{v}_3$ are always small for data on the blue sheet
  - $\mathbf{v}_3$ likely represents a "noise subspace" for these data

Basis based representation



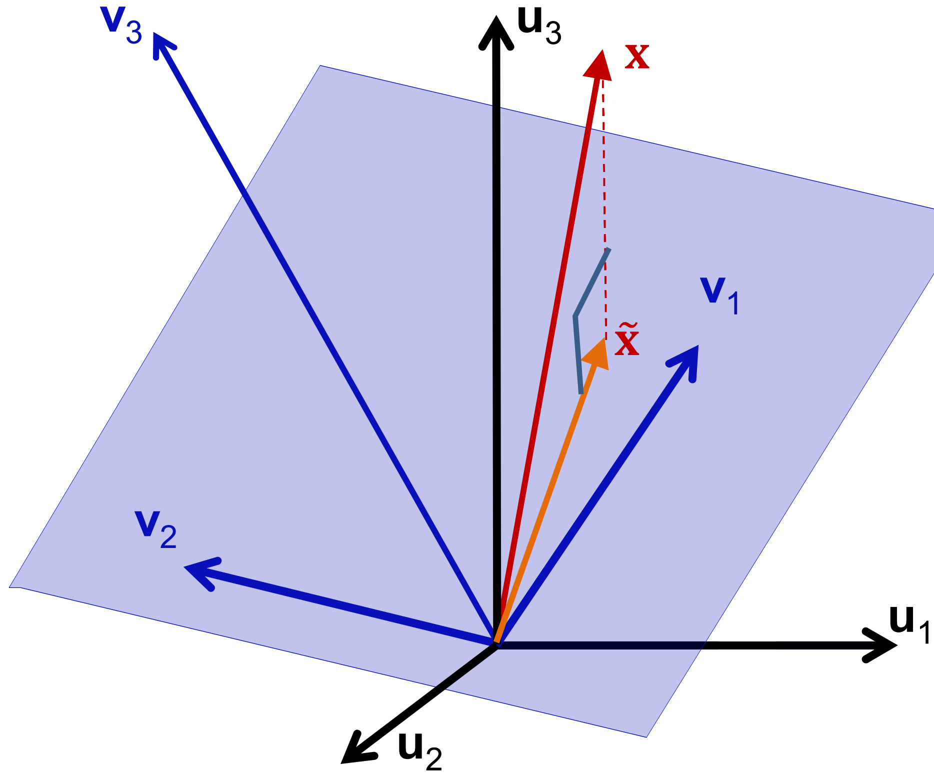
- *The most important challenge* in ML: Find the best set of bases for a given data set

Basis based representation



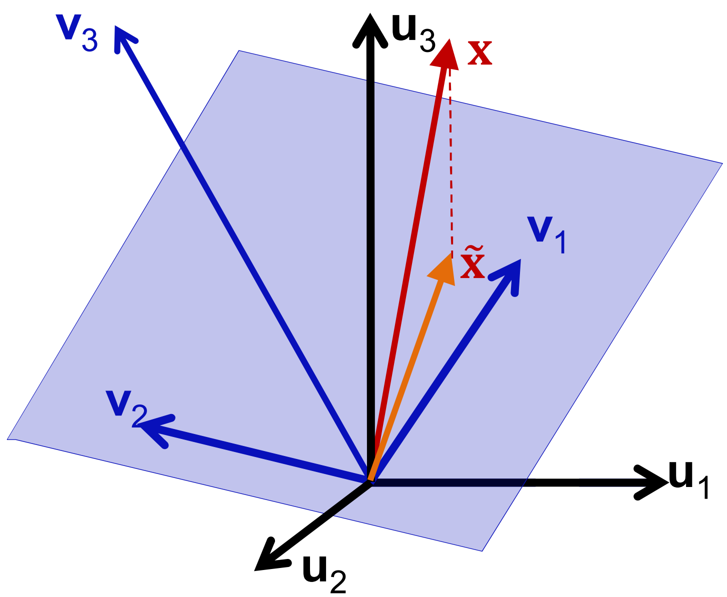
- **Modified problem:** Given the new bases v_1, v_2, v_3
  - Find best representation of every data point on v_1-v_2 plane
    - Put it on the main sheet and disregard the v3 component

Basis based representation



- **Modified problem:**
  - For any vector $\mathbf{x}$
  - Find the closest approximation $\tilde{\mathbf{x}} = a\mathbf{v}_1 + b\mathbf{v}_2$
    - Which lies entirely in the $\mathbf{v}_1$-$\mathbf{v}_2$ plane

Basis based representation



$$\mathbf{V} = [\mathbf{v}_1 \mathbf{v}_2] \qquad \mathbf{a} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\mathbf{x} \approx \mathbf{Va}$$

$$\Downarrow$$

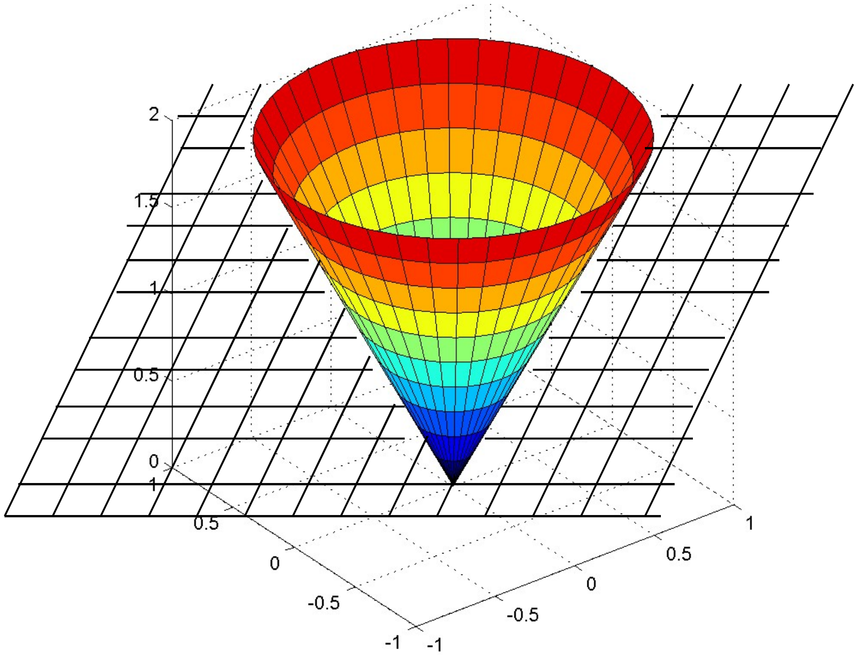
$$\mathbf{a} = \mathbf{V}^{+}\mathbf{x}$$

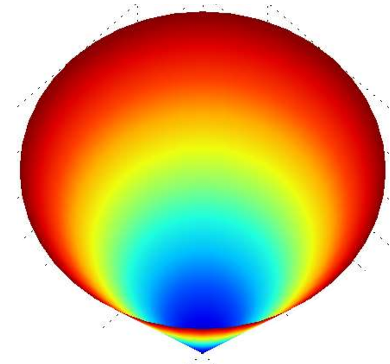
$$\Downarrow$$

$$\tilde{\mathbf{x}} = \mathbf{V}\mathbf{V}^{+}\mathbf{x}$$

- $\mathbf{P} = \mathbf{V}\mathbf{V}^{+}$ is the "projection" matrix that "projects" any vector $\mathbf{x}$ down to its "shadow" $\tilde{\mathbf{x}}$ on the $\mathbf{v}_1$-$\mathbf{v}_2$ plane
  - Expanding: $\mathbf{P} = \mathbf{V}\left(\mathbf{V}^{\mathrm{T}}\mathbf{V}\right)^{-1}\mathbf{V}^{\mathrm{T}}$

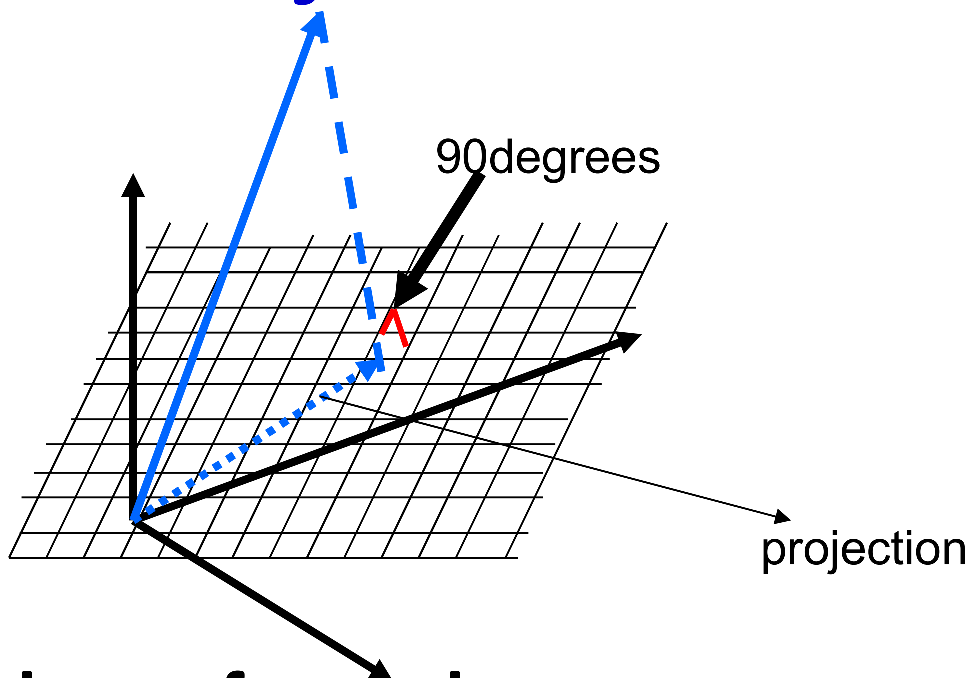
Projections onto a plane





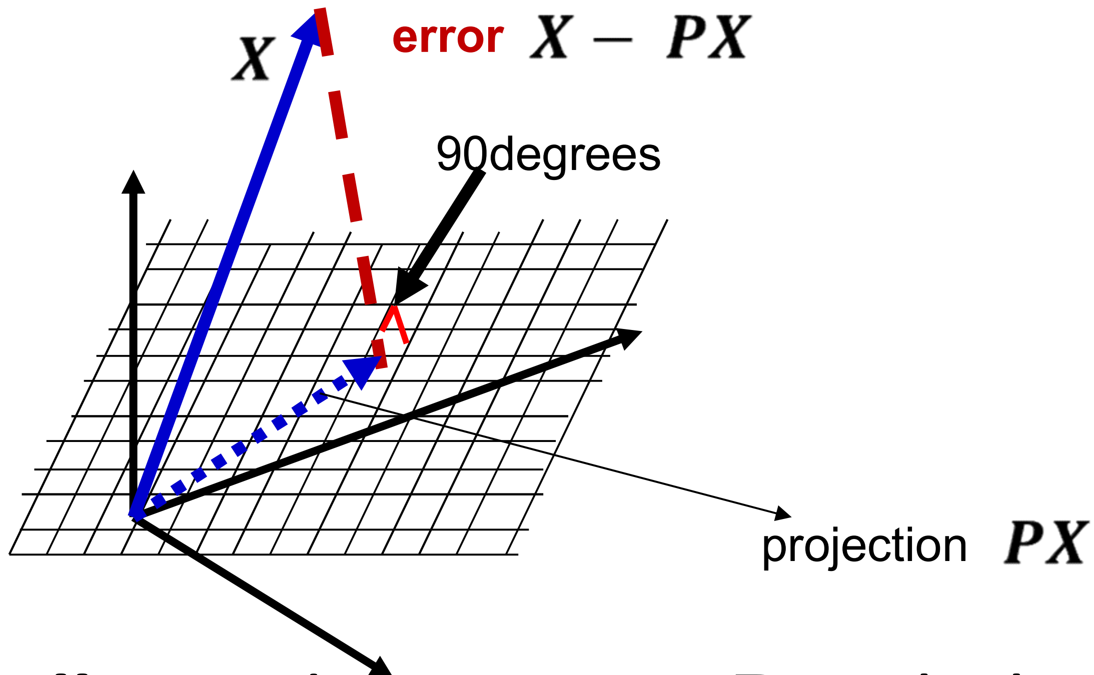
- What would we see if the cone to the left were transparent if we looked at it from above the plane shown by the grid?
  - Normal to the plane
  - Answer: the figure to the right
- How do we get this? Projection

Projections



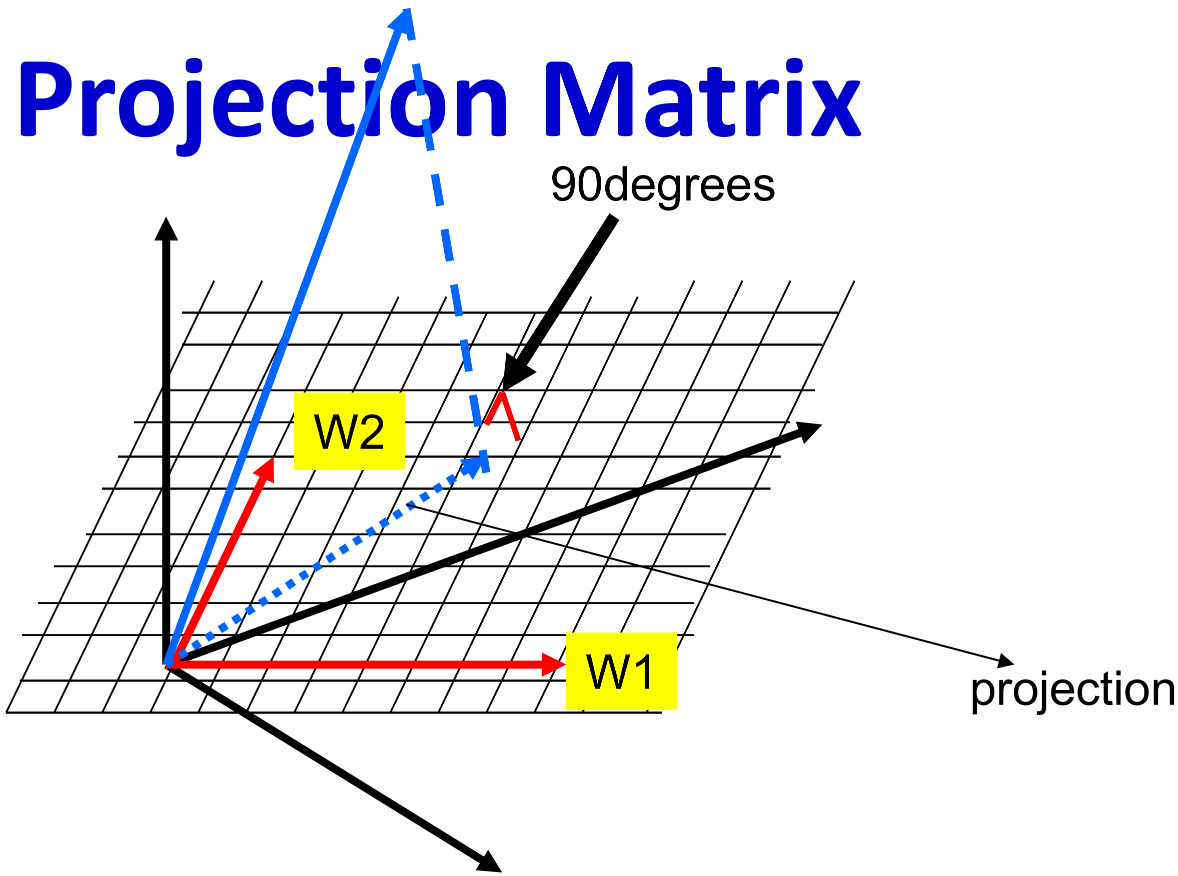
- **Actual problem: for each vector**
  - What is the corresponding vector on the plane that is “closest approximation” to it?
  - What is the *transform* that converts the vector to its approximation on the plane?

Projections



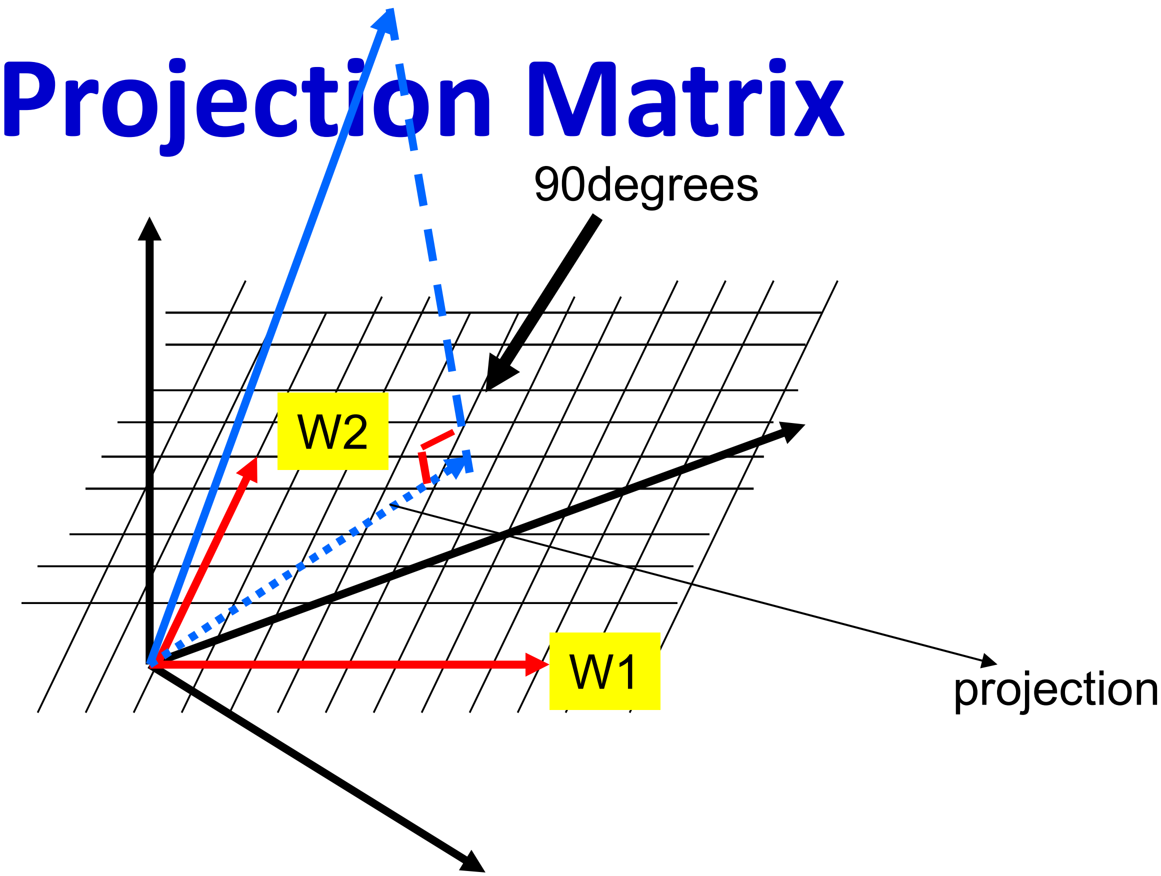
- **Arithmetically:** Find the matrix $\boldsymbol{P}$ such that
  - For **every** vector $\boldsymbol{X}$, $\boldsymbol{PX}$ lies on the plane
    - The plane is the column space of $\boldsymbol{P}$
  - $||X - \boldsymbol{P}X||^2$ is the smallest possible

Projection Matrix



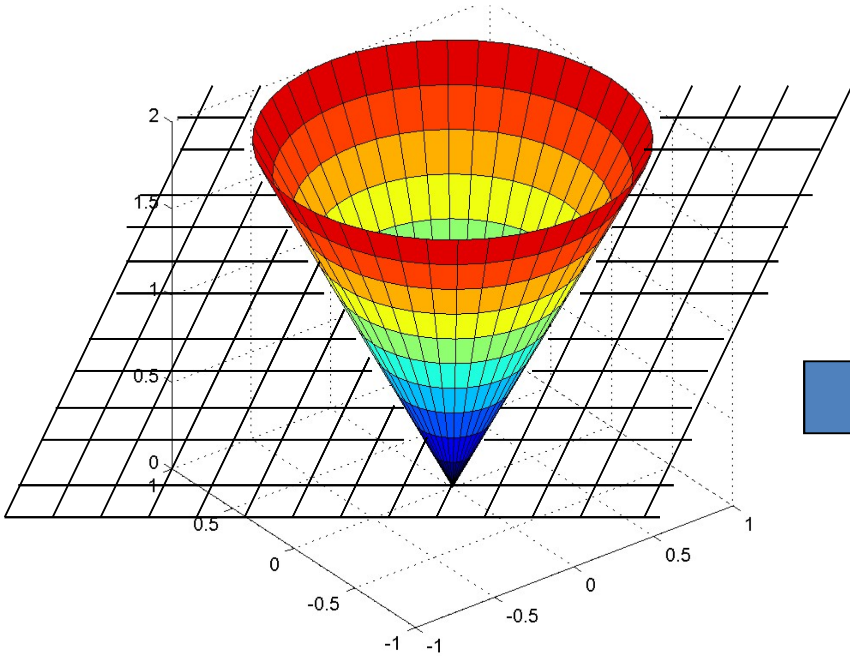
- Consider any set of *independent* vectors (bases) $\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots$ on the plane
  - Arranged as a matrix $[\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots]$
  - The plane is the *column space* of the matrix
- Find the projection matrix $\boldsymbol{P}$ that projects on to the plane formed from $[\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots]$

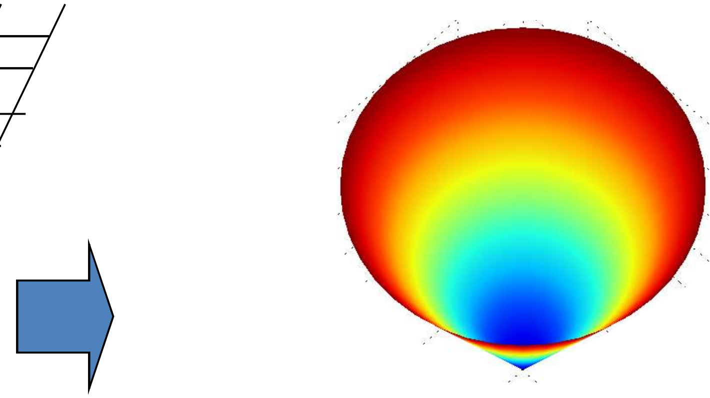
Projection Matrix



- Given a set of vectors $\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots$ which form a matrix $\boldsymbol{W} = [\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots]$
- The projection matrix to transform a vector $\boldsymbol{X}$ to its projection on the plane is
  - $\boldsymbol{P} = \boldsymbol{W}(\boldsymbol{W}^T\boldsymbol{W})^{-1}\boldsymbol{W}^T$

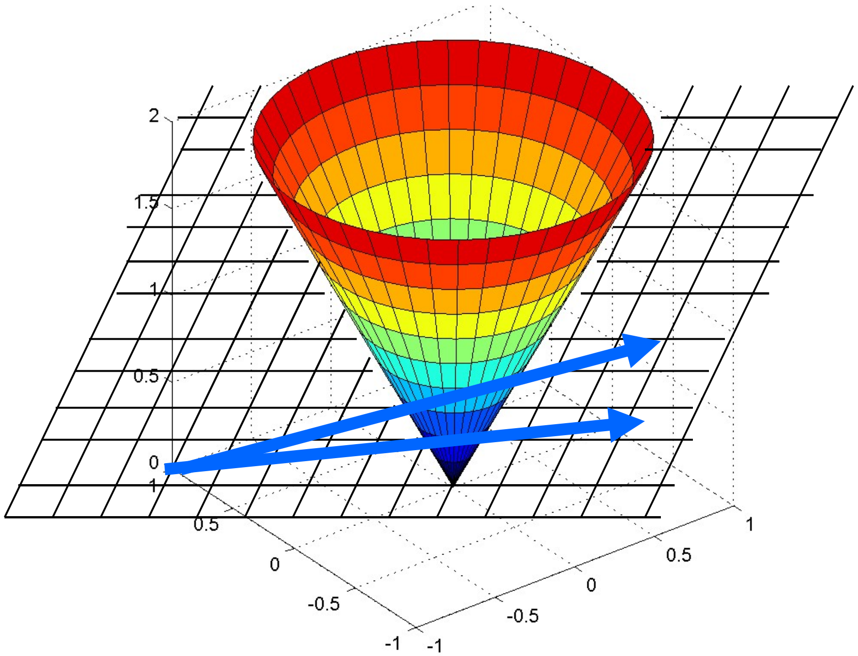
Projections

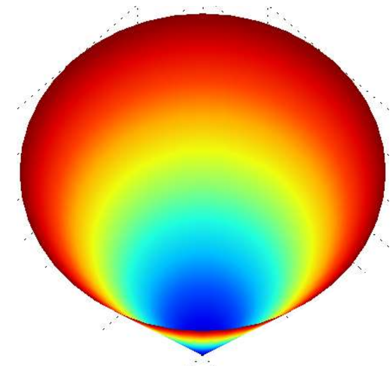




- HOW?

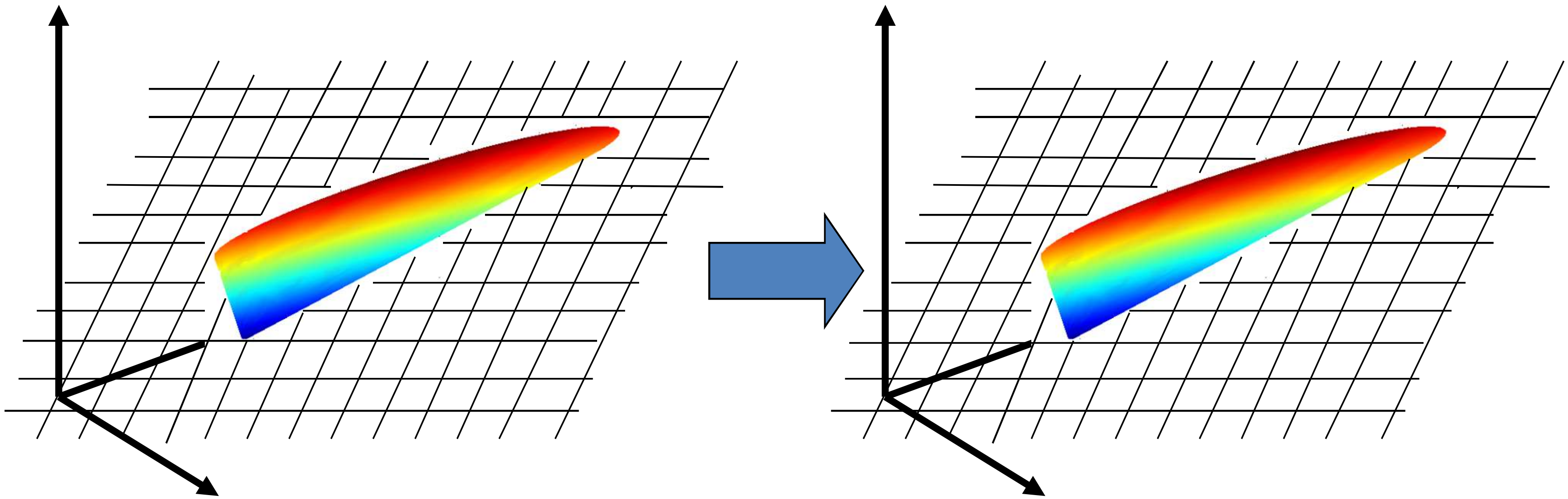
Projections





- Draw any two vectors W_1 and W_1 W_2 that lie on the plane
  - ***ANY two*** **so long as they have different angles**
- Compose a matrix $\mathbf{W} = [W_1\ W_2 ..]$
- Compose the projection matrix $\mathbf{P} = \mathbf{W}\ (\mathbf{W}^T\mathbf{W})^{-1}\ \mathbf{W}^T$
- Multiply every point on the cone by $\mathbf{P}$ to get its projection

Projection matrix properties



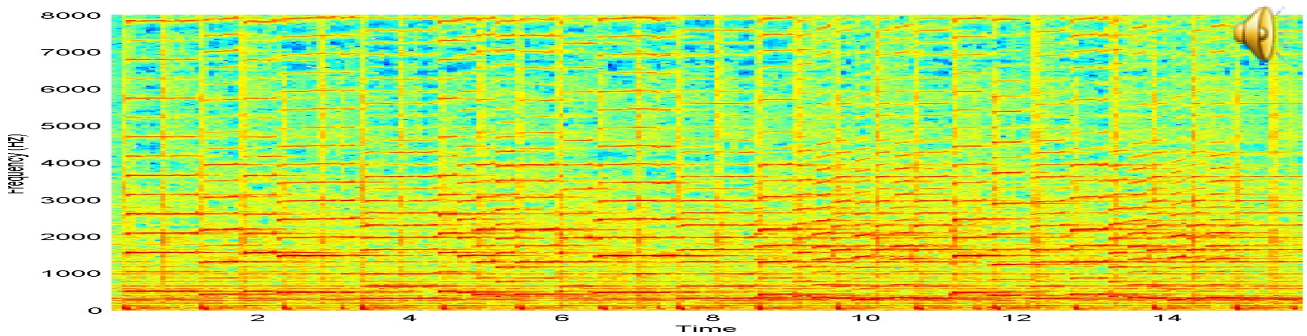
- The projection of any vector that is already on the plane is the vector itself
  - $\mathbf{PX} = \mathbf{X}$ if $\mathbf{X}$ is on the plane
  - If the object is already on the plane, there is no further projection to be performed
- The projection of a projection is the projection
  - $\mathbf{P}(\mathbf{PX}) = \mathbf{PX}$
- Projection matrices are *idempotent*
  - $\mathbf{P}^2 = \mathbf{P}$

Projections: A more physical meaning

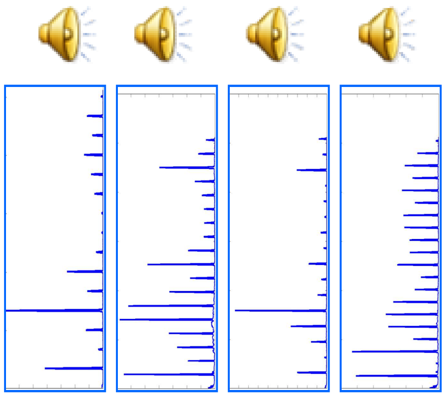
- Let $\mathbf{W}_1$, $\mathbf{W}_2$ .. $\mathbf{W}_k$ be "bases"
- We want to explain our data in terms of these "bases"
  - We often cannot do so
  - But we can explain a significant portion of it

- The portion of the data that can be expressed in terms of our vectors $\mathbf{W}_1$, $\mathbf{W}_2$, ... $\mathbf{W}_k$, is the projection of the data on the $\mathbf{W}_1$ … $\mathbf{W}_k$ (hyper) plane
  - In our previous example, the "data" were all the points on a cone, and the bases were vectors on the plane

Projection : an example with sounds

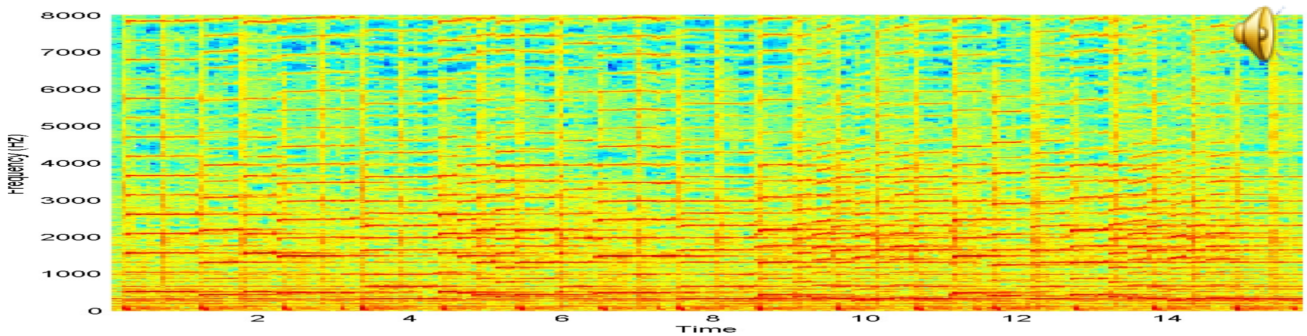


- The spectrogram (matrix) of a piece of music

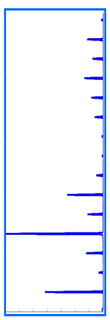


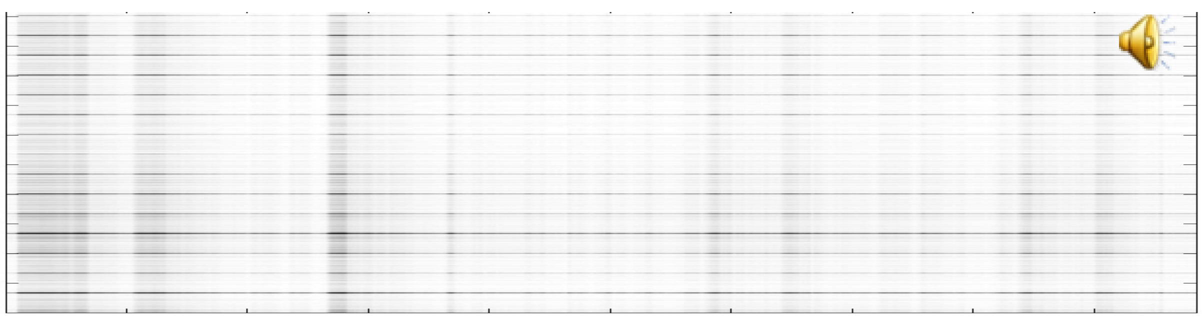
- How much of the above music was composed of the above notes
  - I.e. how much can it be explained by the notes

Projection: one note

M = 

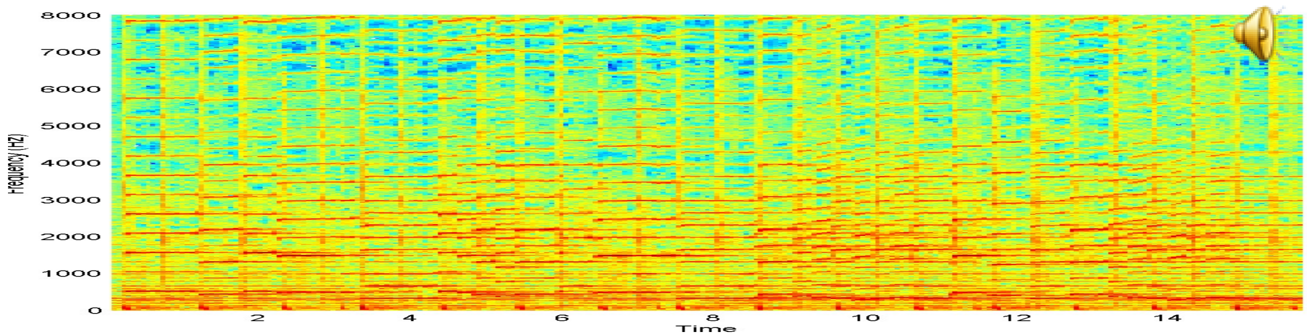
- The spectrogram (matrix) of a piece of music

W = 

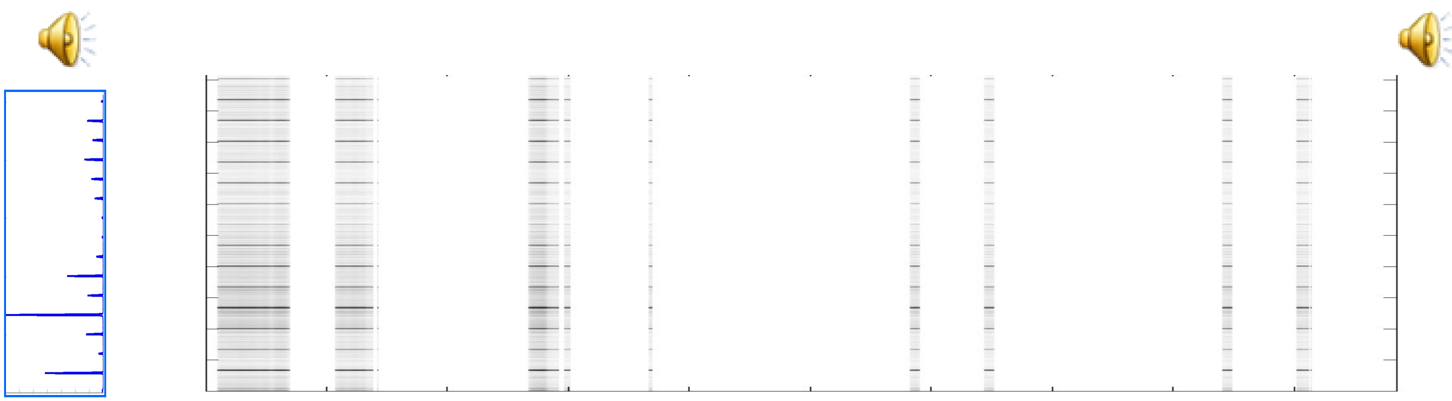


- M = spectrogram; W = note
- $P = W(W^TW)^{-1}W^T$
- Projected Spectrogram = PM

Projection: one note – cleaned up

M = 

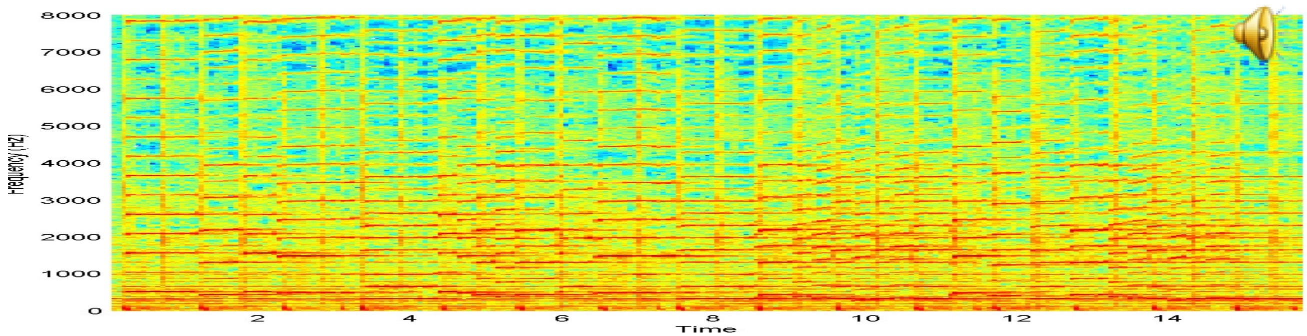
- The spectrogram (matrix) of a piece of music

W = 

- Floored all matrix values below a threshold to zero

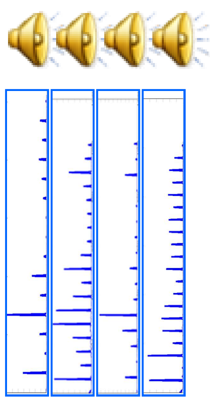
Projection: multiple notes

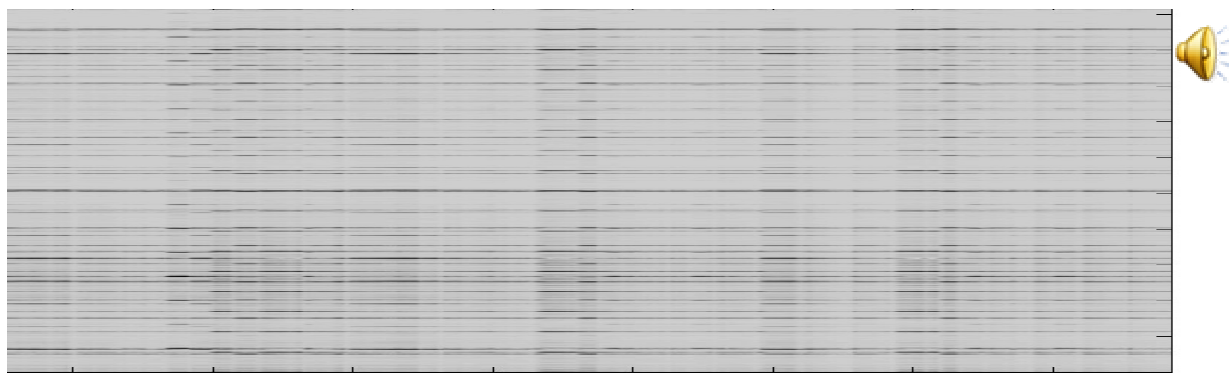
M =



- The spectrogram (matrix) of a piece of music

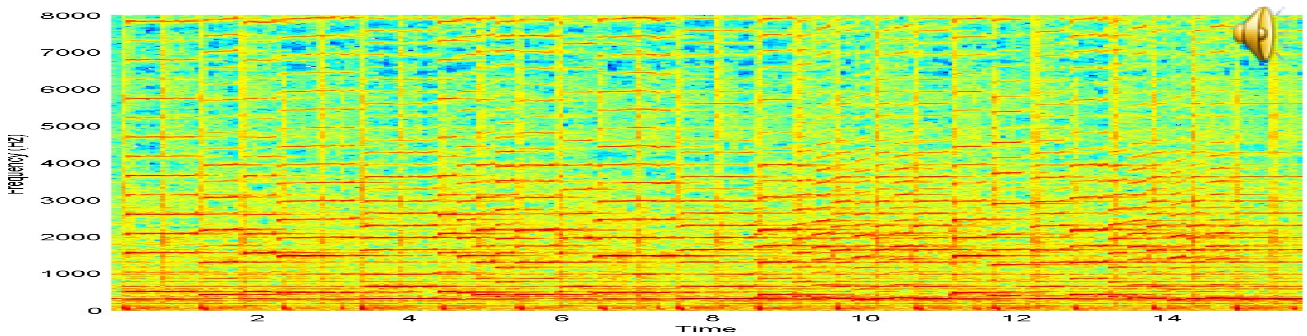
W =



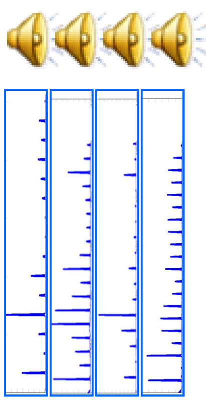


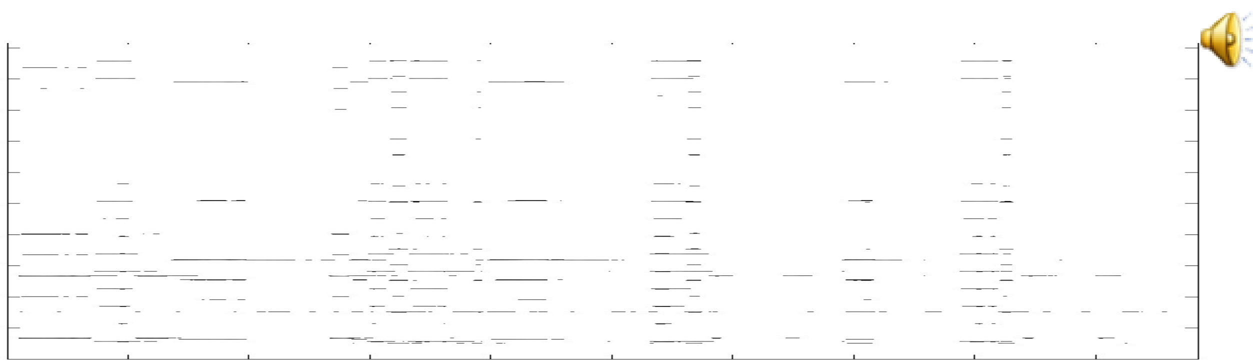
- $P = W (W^T W)^{-1} W^T$
- Projected Spectrogram = P * M

Projection: multiple notes, cleaned up

M = 

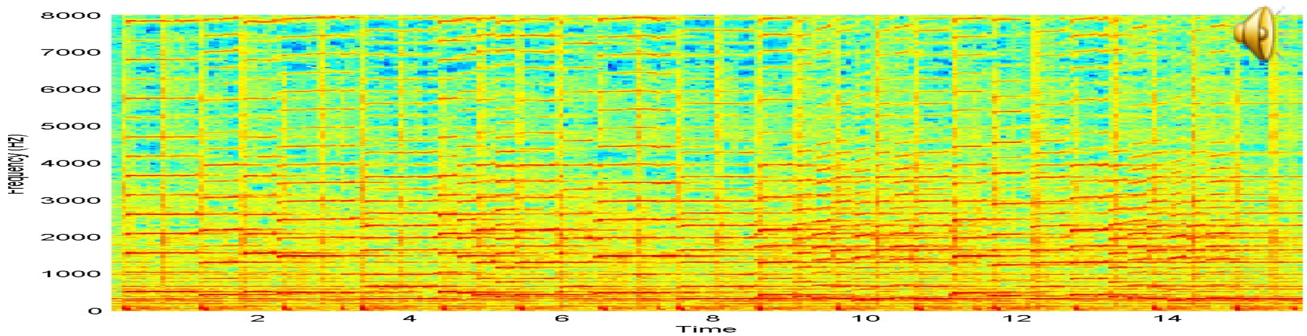
- The spectrogram (matrix) of a piece of music

W = 



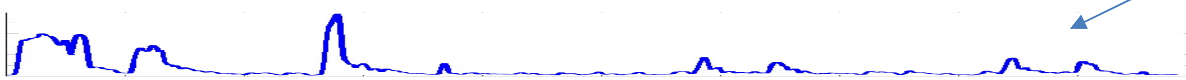
- $\boldsymbol{P} = \boldsymbol{W}(\boldsymbol{W}^T\boldsymbol{W})^{-1}\boldsymbol{W}^T$
- Projected Spectrogram = $\boldsymbol{PM}$

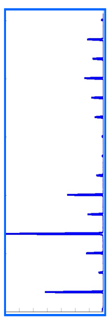
Projection: one note

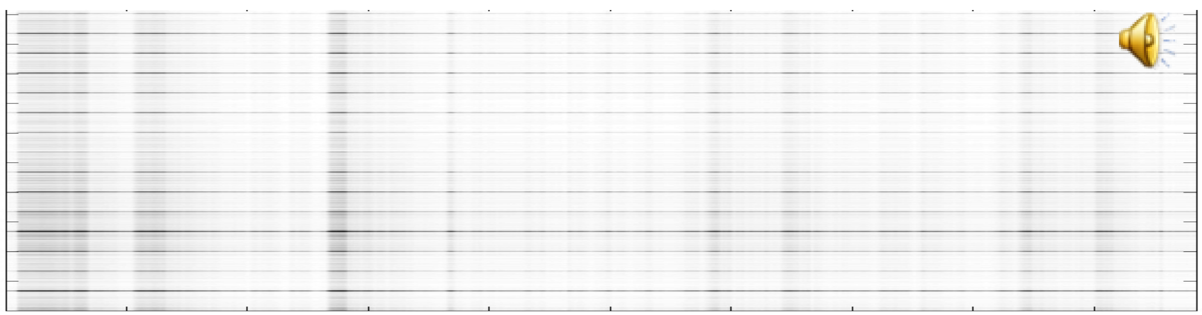
M = 

- The spectrogram (matrix) of a piece of music

$T = W^{+}M$



W = 



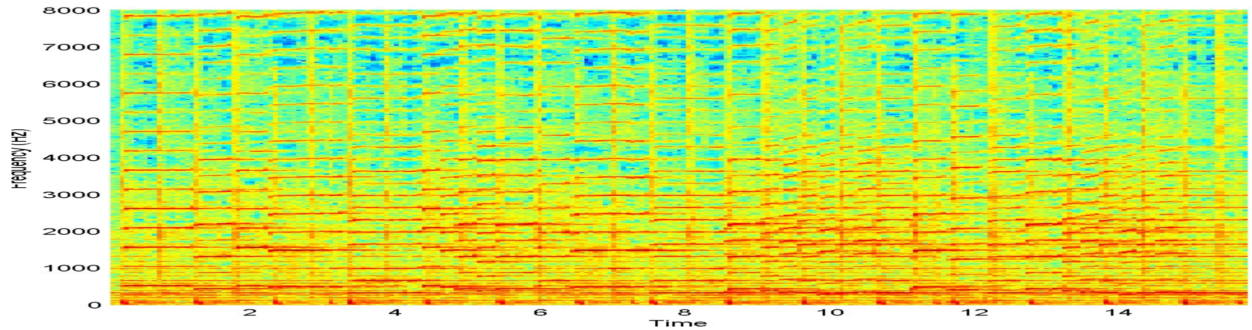
- The "transcription" of the note is

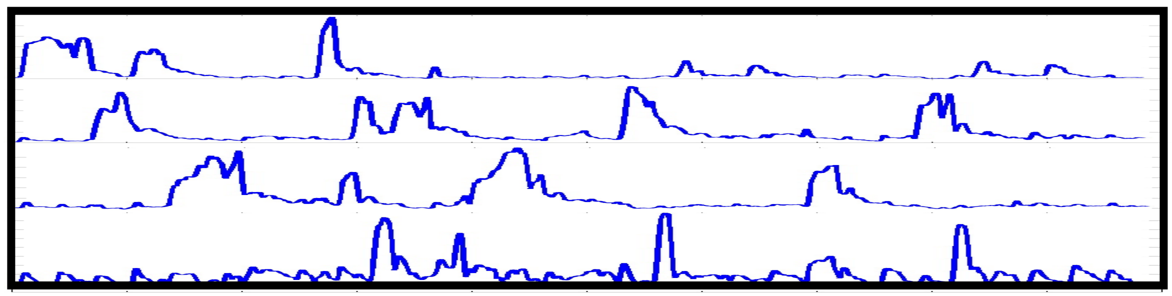
$$T = W^{+}M = (W^{T}W)^{-1}W^{T}M$$

- Projected Spectrogram = $WT = PM$

Explanation with multiple notes

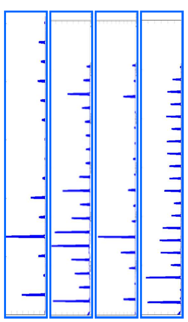
M =

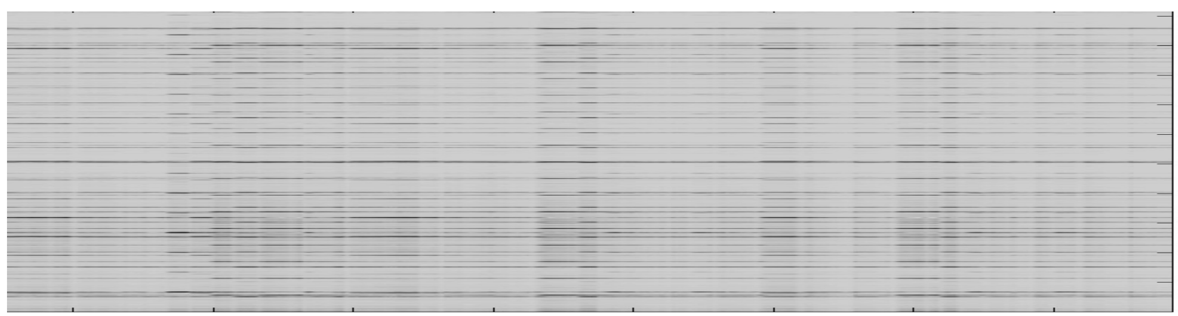




$T = W^{+}M$

W =





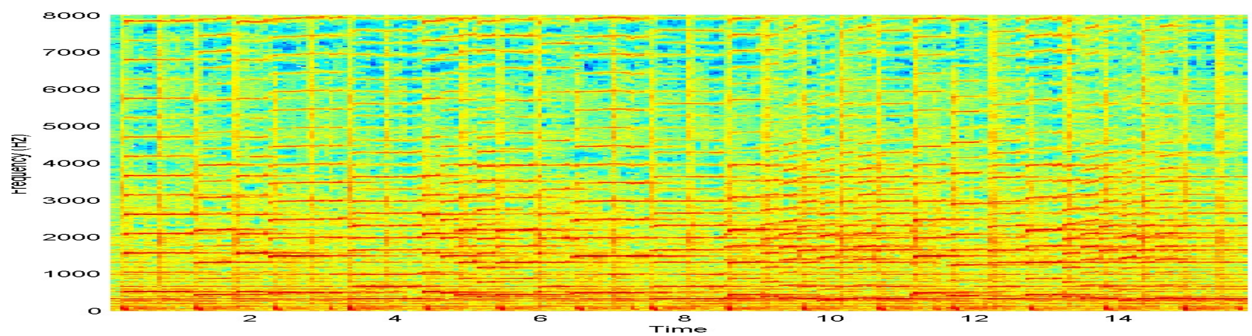
- The "transcription" of the set of notes is

$$T = W^{+}M = (W^{T}W)^{-1}W^{T}M$$

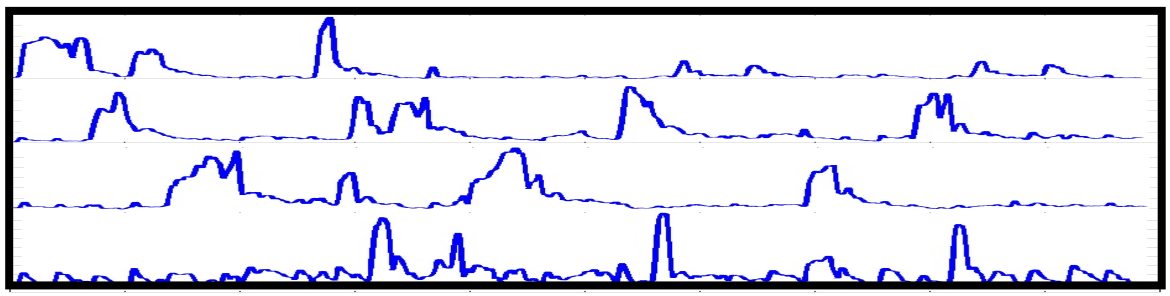
- Projected Spectrogram = $WT = PM$

How about the other way?

M =



T =

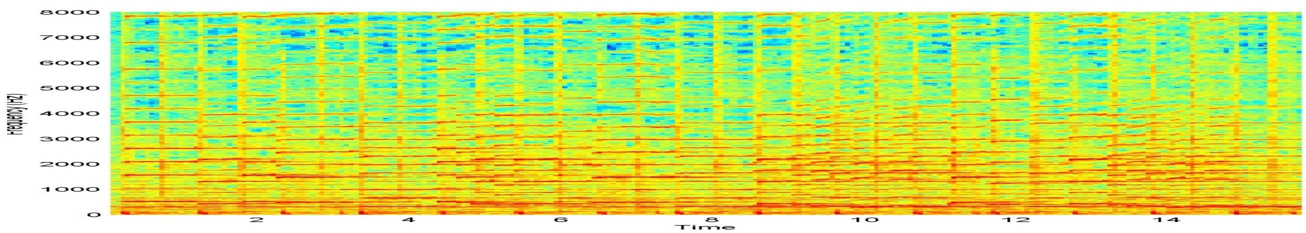


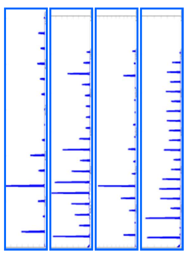
W = ?

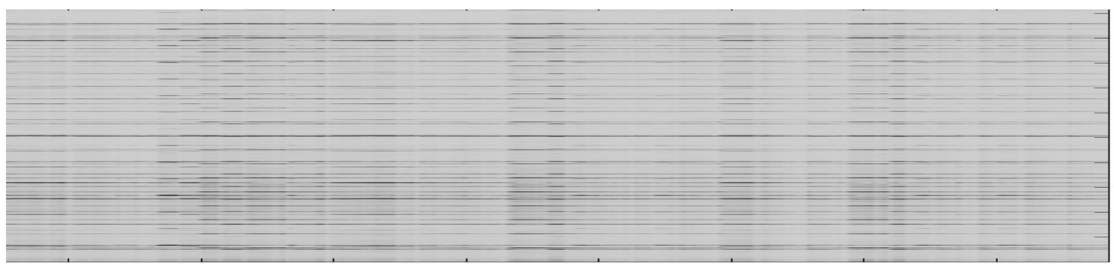
U = ?

- $WT \approx M$ $W = M\,Pinv(T)$ $U = WT$

Projections are often examples of rank-deficient transforms

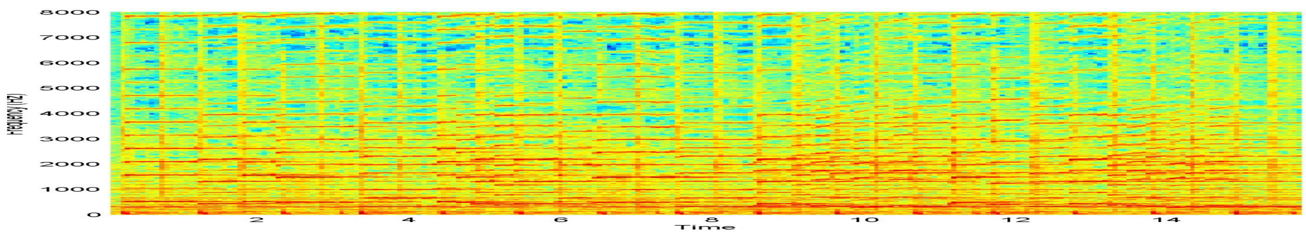
M = 

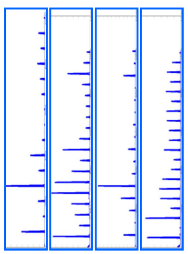
W = 

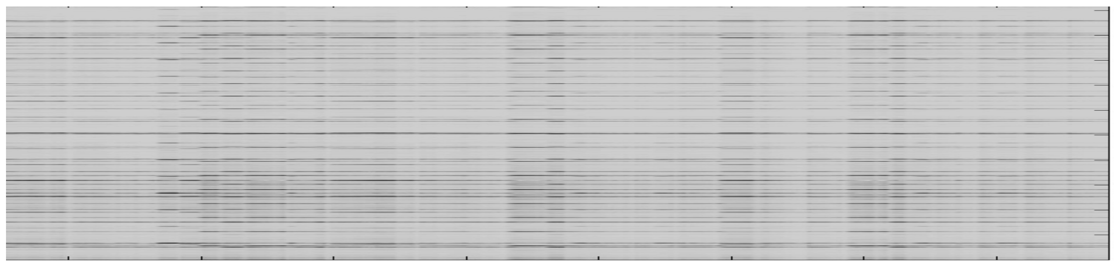


- $P = W(W^T W)^{-1} W^T$; Projected Spectrogram : $M_{proj} = PM$
- The original spectrogram can never be recovered
  - P is rank deficient
- P explains all vectors in the new spectrogram as a mixture of only the 4 vectors in W
  - There are only a maximum of 4 ***linearly independent*** bases
  - Rank of P is 4

The Rank of Matrix

M = 





- Projected Spectrogram = $P\ M$
  - Every vector in it is a combination of only 4 bases
- The rank of the matrix is the *smallest* no. of bases required to describe the output
  - E.g. if note no. 4 in P could be expressed as a combination of notes 1,2 and 3, it provides no additional information
  - Eliminating note no. 4 would give us the same projection
  - The rank of P would be 3!

Pseudo-inverse (PINV)

- $Pinv()$ applies to non-square matrices and non-invertible square matrices
- $Pinv\ (Pinv(\mathbf{A}))\ =\ \mathbf{A}$
- $\mathbf{A}Pinv(\mathbf{A})$= projection matrix!
  - Projection onto the columns of $\mathbf{A}$

- If $\mathbf{A}$ is a $K \times N$ matrix and $K > N$, $\mathbf{A}$ projects N-dimensional vectors into a higher-dimensional K-dimensional space
  - $Pinv(\mathbf{A})$ is a $N \times K$ matrix
  - $Pinv(\mathbf{A})\mathbf{A} = \mathbf{I}$ in this case
- Otherwise $\mathbf{A}Pinv(\mathbf{A}) = \mathbf{I}$

Overview

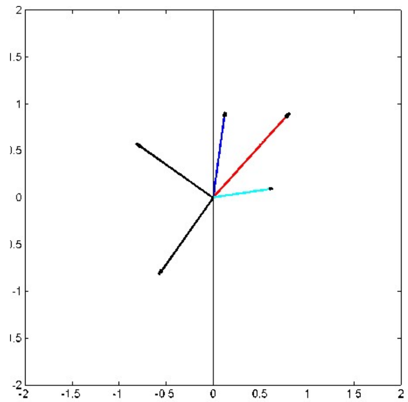
- Vectors and matrices
- Basic vector/matrix operations
- Various matrix types
- Matrix properties
  - Determinant
  - Inverse
  - Rank
- Solving simultaneous equations
- Projections
- **Eigen decomposition**
- **SVD**

Eigenanalysis

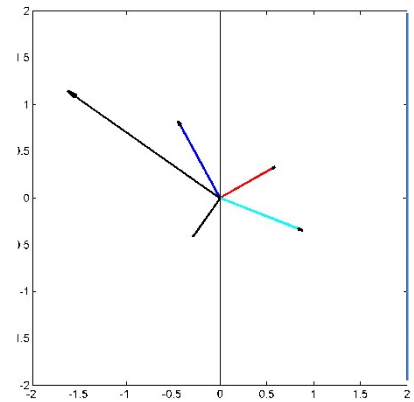
- If something can go through a process mostly unscathed in character it is an *eigen*-something
  - Sound example:
- A vector that can undergo a matrix multiplication and keep pointing the same way is an *eigenvector*
  - Its length can change though
- How much its length changes is expressed by its corresponding *eigenvalue*
  - Each eigenvector of a matrix has its eigenvalue
- Finding these “eigenthings” is called eigenanalysis

EigenVectors and EigenValues

Black vectors are eigen vectors



$$M = \begin{bmatrix} 1.5 & -0.7 \\ -0.7 & 1.0 \end{bmatrix}$$

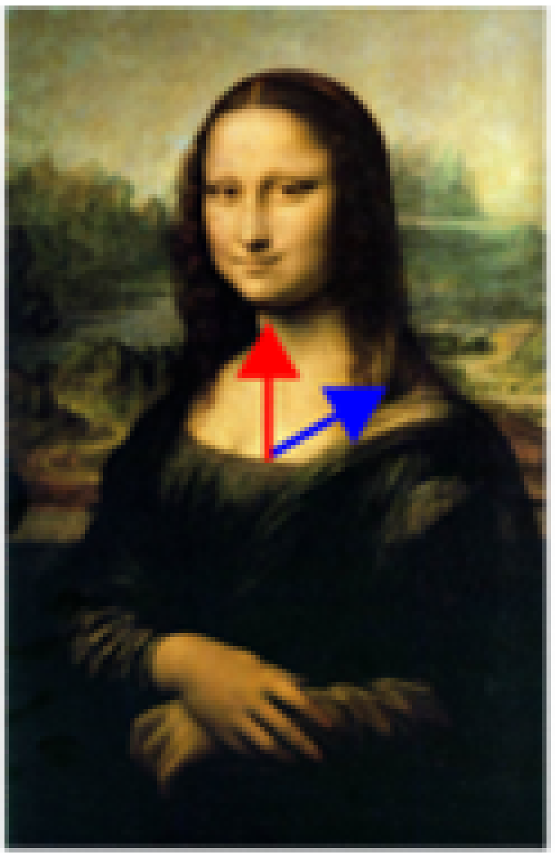


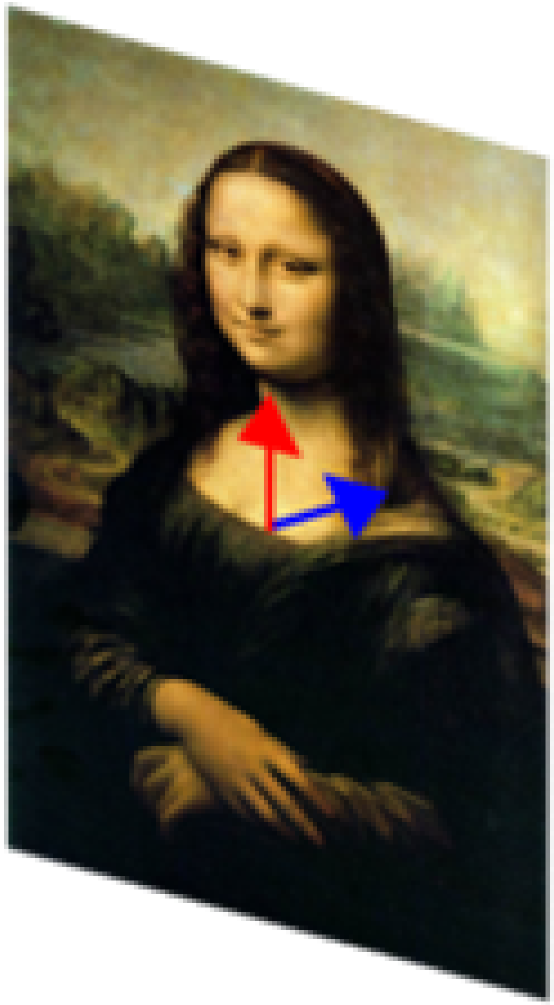
- Vectors that do not change angle upon transformation
  - They may change length

$$MV = \lambda V$$

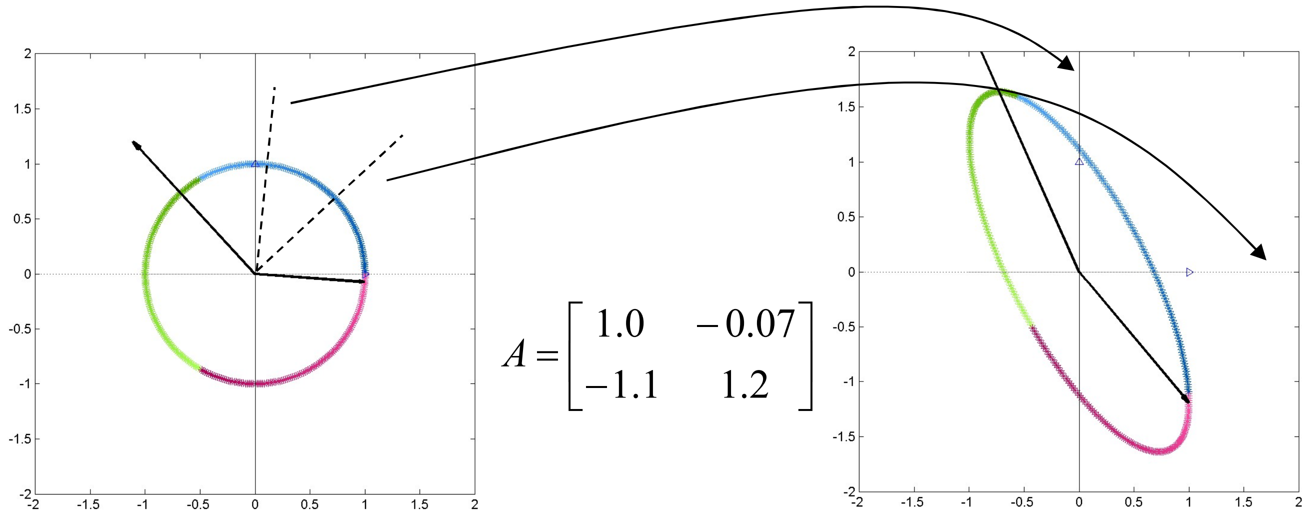
  - V = eigen vector
  - λ = eigen value

Eigen vector example



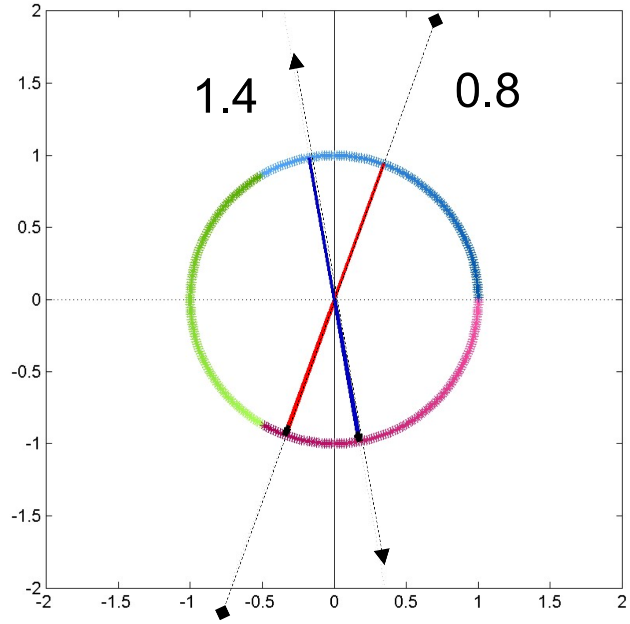


Matrix multiplication revisited



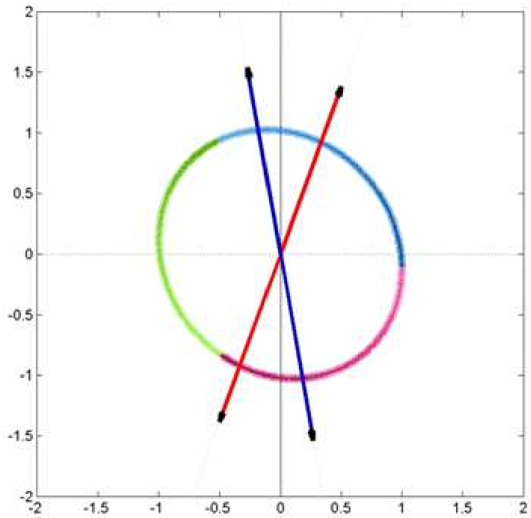
- Matrix transformation “transforms” the space
  - Warps the paper so that the normals to the two vectors now lie along the axes

A stretching operation



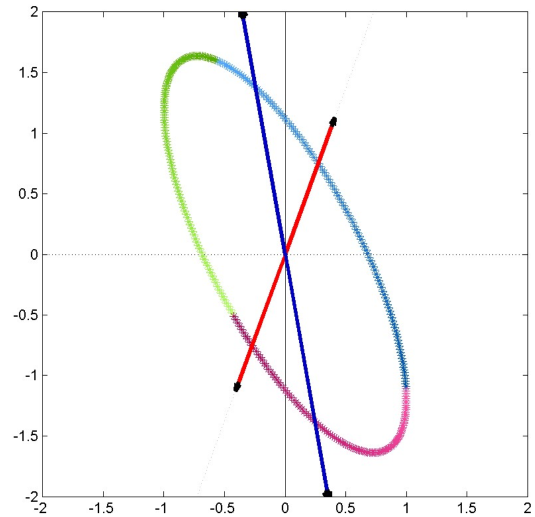
- Draw two lines
- Stretch / shrink the paper along these lines by factors λ_1 and λ_2
  - The factors could be negative – implies flipping the paper
- The result is a transformation of the space

A stretching operation



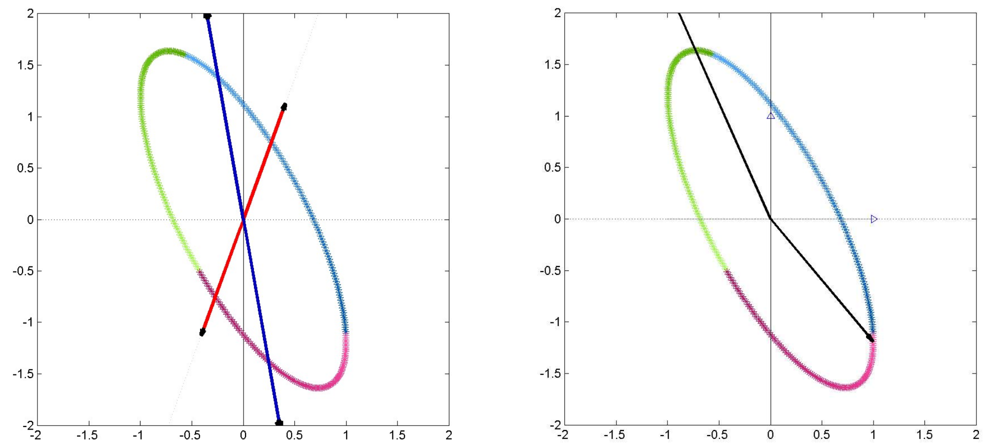
- Draw two lines
- Stretch / shrink the paper along these lines by factors λ_1 and λ_2
  - The factors could be negative – implies flipping the paper
- The result is a transformation of the space

A stretching operation



- Draw two lines
- Stretch / shrink the paper along these lines by factors λ_1 and λ_2
  - The factors could be negative – implies flipping the paper
- The result is a transformation of the space

Physical interpretation of eigen vector



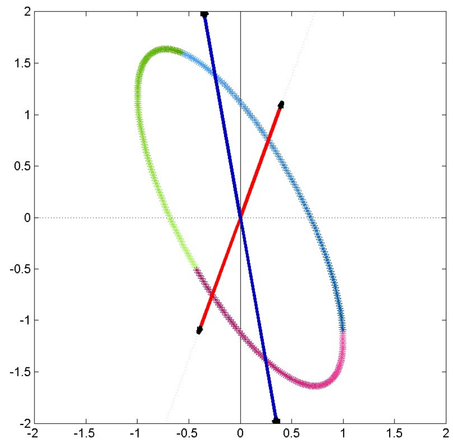
- The result of the stretching is exactly the same as transformation by a matrix
- The axes of stretching/shrinking are the eigenvectors
  - The degree of stretching/shrinking are the corresponding eigenvalues
- The EigenVectors and EigenValues convey all the information about the matrix

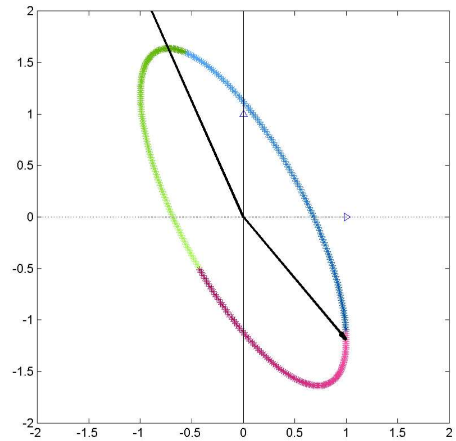
Physical interpretation of eigen vector

$$V = [V_1 \quad V_2]$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$M = V \Lambda V^{-1}$$





- The result of the stretching is exactly the same as transformation by a matrix
- The axes of stretching/shrinking are the eigenvectors
  - The degree of stretching/shrinking are the corresponding eigenvalues
- The EigenVectors and EigenValues convey all the information about the matrix
- The determinant of the matrix is the product of the eigenvalues

$$|M| = |V||\Lambda||V^{-1}| = C.\prod_i \lambda_i \, .C^{-1} = \prod_i \lambda_i$$

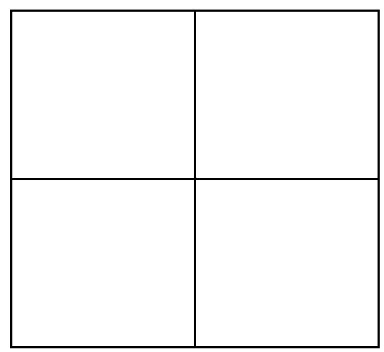
Eigen Analysis

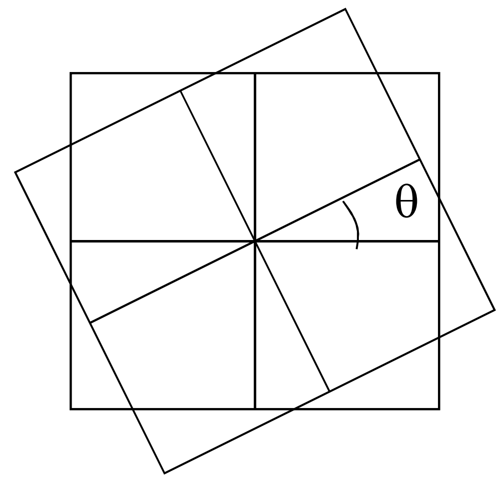
- Not all square matrices have nice eigen values and vectors
  - E.g. consider a rotation matrix

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$X_{new} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

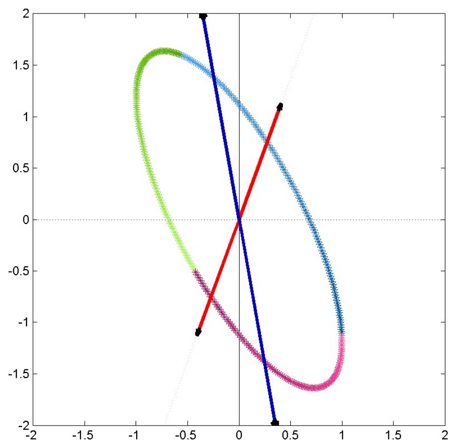




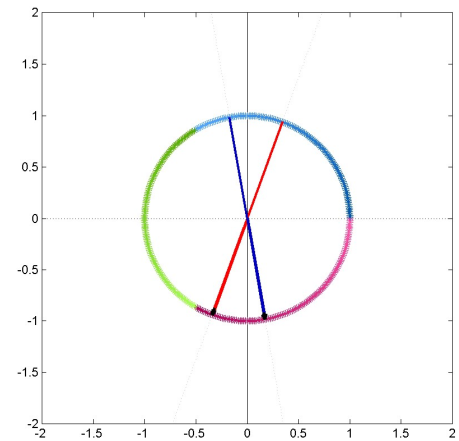
  - This rotates every vector in the plane
    - No vector that remains unchanged

- In these cases the Eigen vectors and values are complex
  - Actually complex conjugate pairs

Singular Value Decomposition

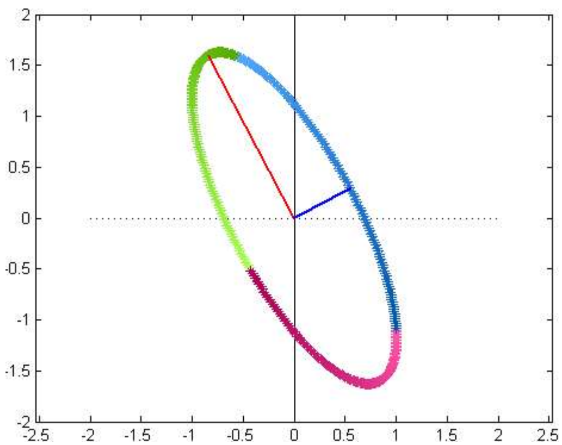


$$A = \begin{bmatrix} 1.0 & -0.07 \\ -1.1 & 1.2 \end{bmatrix}$$

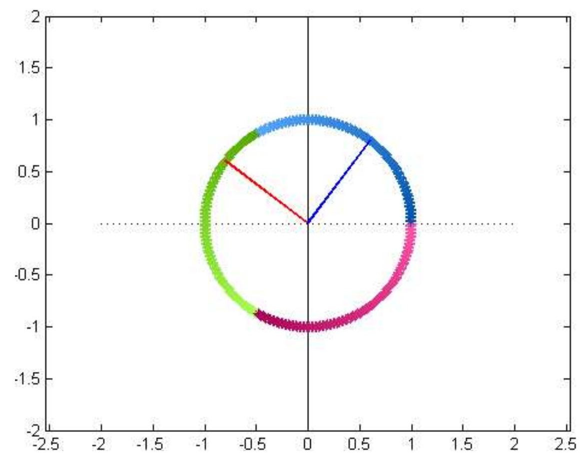


- Matrix transformations convert circles to ellipses
- Eigen vectors are vectors that do not change direction in the process
- There is another key feature of the ellipse to the left that carries information about the transform
  - Can you identify it?

Singular Value Decomposition

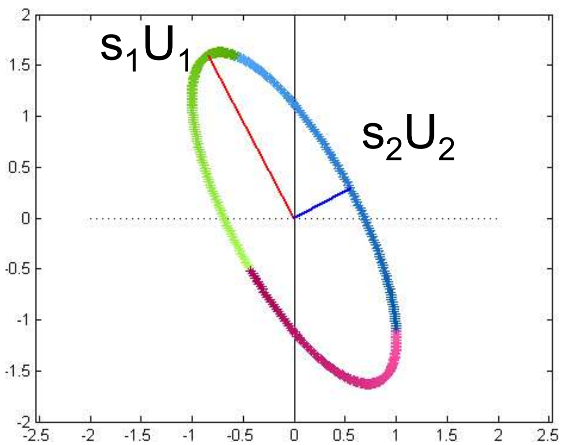


$$A = \begin{bmatrix} 1.0 & -0.07 \\ -1.1 & 1.2 \end{bmatrix}$$



- The major and minor axes of the transformed ellipse define the ellipse
  - They are at right angles
- These are transformations of right-angled vectors on the original circle!

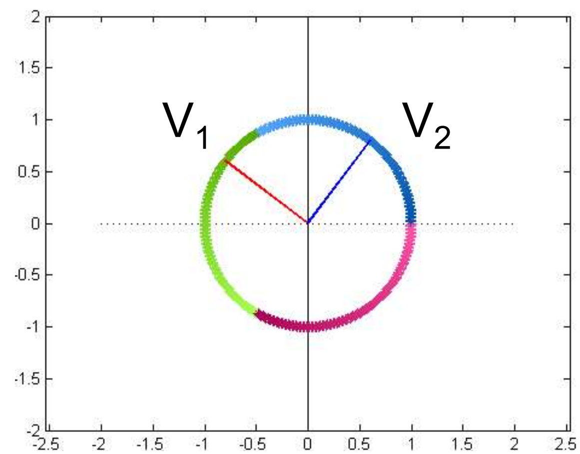
Singular Value Decomposition



$$A = \begin{bmatrix} 1.0 & -0.07 \\ -1.1 & 1.2 \end{bmatrix}$$

$$A = U\ S\ V^T$$

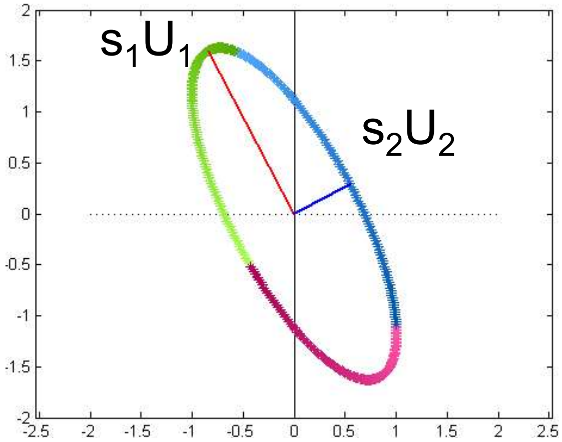
matlab:
[U,S,V] = svd(A)



- U and V are orthonormal matrices
  - Columns are orthonormal vectors
- S is a diagonal matrix

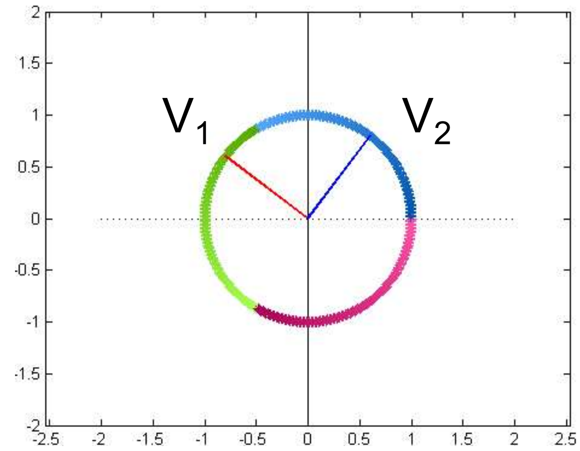
- The *right singular vectors* in V are transformed to the *left singular vectors* in U
  - And scaled by the *singular values* that are the diagonal entries of S

Singular Value Decomposition



$A = U S V^T$

$A^T = V S U^T$

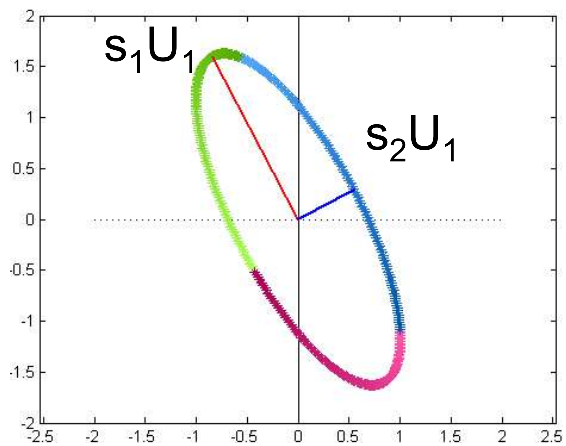


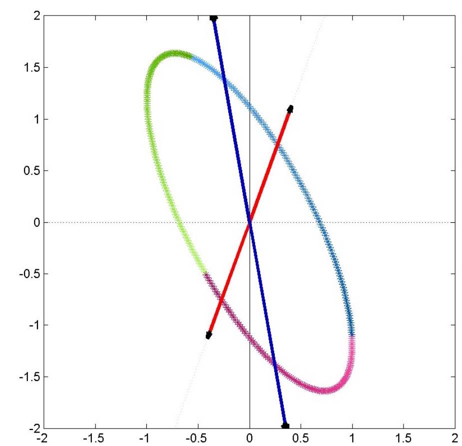
- A matrix $\boldsymbol{A}$ converts *right* singular vectors $\boldsymbol{V}$ to *left* singular vectors $\boldsymbol{U}$
- $\boldsymbol{A}^{\mathrm{T}}$ converts $\boldsymbol{U}$ to $\boldsymbol{V}$

Singular Value Decomposition

- The left and right singular vectors are not the same
  - If A is not a square matrix, the left and right singular vectors will be of different dimensions
- The singular values are always real
- The largest singular value is the largest amount by which a vector is scaled by A
  - Max (|Ax| / |x|) = s_{max}
- The smallest singular value is the smallest amount by which a vector is scaled by A
  - Min (|Ax| / |x|) = s_{min}
  - This can be 0 (for low-rank or non-square matrices)

The Singular Values

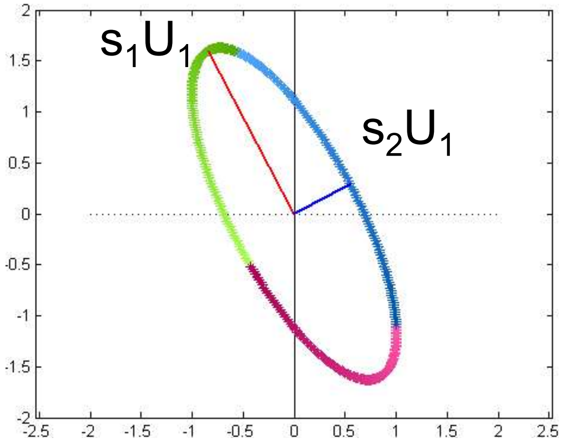


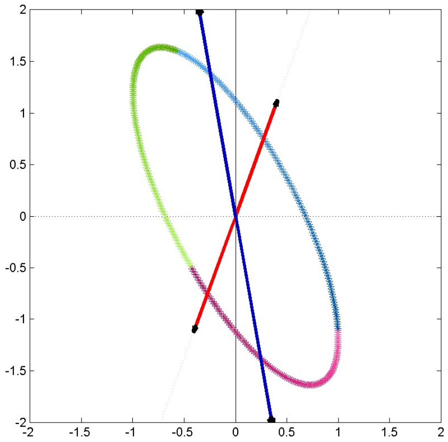


- Square matrices: product of singular values = determinant of the matrix
  - This is also the product of the *eigen* values
  - I.e. there are two different sets of axes whose products give you the area of an ellipse

- For any "broad" rectangular matrix A, the largest singular value of any square submatrix B cannot be larger than the largest singular value of A
  - An analogous rule applies to the smallest singular value
  - This property is utilized in various problems

SVD vs. Eigen Analysis

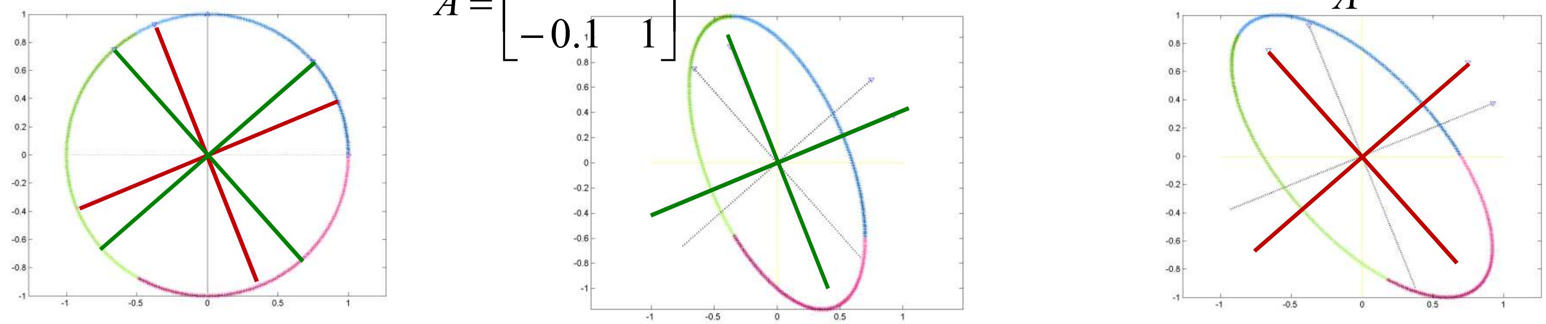




- Eigen analysis of a matrix **A**:
  - Find vectors such that their absolute directions are not changed by the transform
- SVD of a matrix **A**:
  - Find orthogonal set of vectors such that the *angle* between them is not changed by the transform
- For one class of matrices, these two operations are the same

A matrix vs. its transpose

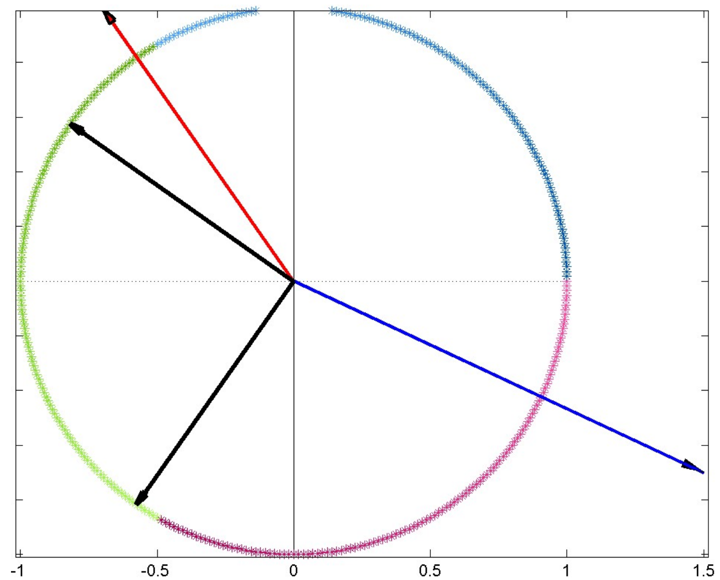
$$A = \begin{bmatrix} .7 & 0 \\ -0.1 & 1 \end{bmatrix} \qquad A^T$$



- Multiplication by matrix A:
  - Transforms right singular vectors in V to left singular vectors U
- Multiplication by its transpose A^T:
  - Transforms *left* singular vectors U to right singular vector V
- $A A^T$: Converts V to U, then brings it back to V
  - Result: Only scaling

Symmetric Matrices

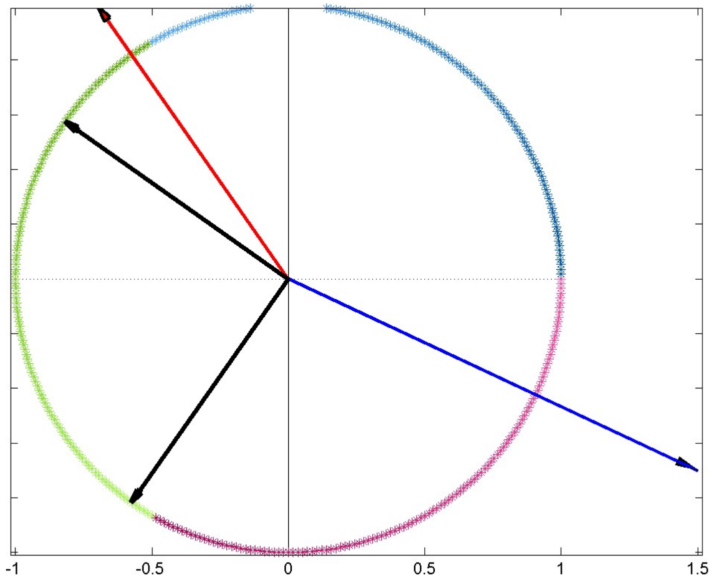
$$\begin{bmatrix} 1.5 & -0.7 \\ -0.7 & 1 \end{bmatrix}$$



- Matrices that do not change on transposition
  - Row and column vectors are identical
- The left and right singular vectors are identical
  - U = V
  - $A = U S U^T$
- They are identical to the *Eigen vectors* of the matrix
- Symmetric matrices do not rotate the space
  - Only scaling and, if Eigen values are negative, reflection

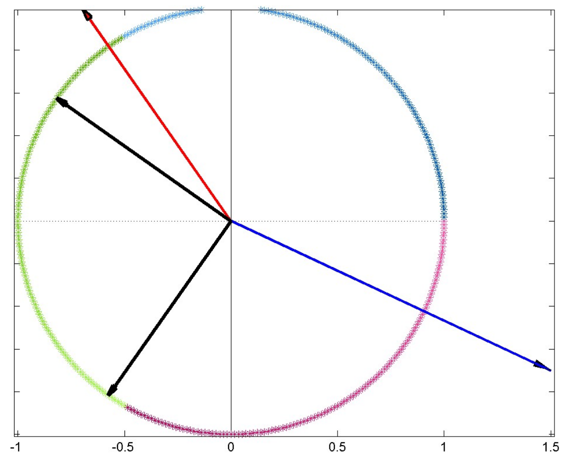
Symmetric Matrices

$$\begin{bmatrix} 1.5 & -0.7 \\ -0.7 & 1 \end{bmatrix}$$

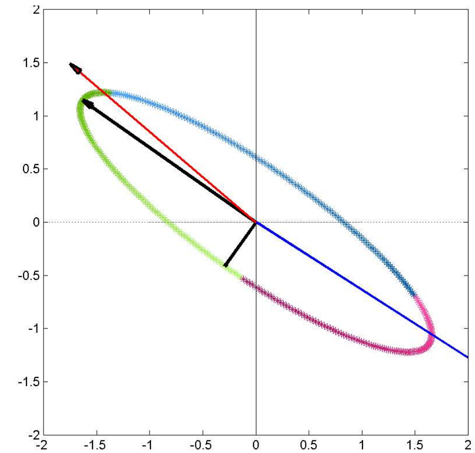


- Matrices that do not change on transposition
  - Row and column vectors are identical
- Symmetric matrix: Eigen vectors and Eigen values are always real
- Eigen vectors are always orthogonal
  - At 90 degrees to one another

Symmetric Matrices



$$\begin{bmatrix} 1.5 & -0.7 \\ -0.7 & 1 \end{bmatrix}$$



- Eigen vectors point in the direction of the major and minor axes of the ellipsoid resulting from the transformation of a spheroid
  - The eigen values are the lengths of the axes

Symmetric matrices

- Eigen vectors V_i are orthonormal
  - $V_i^T V_i = 1$
  - $V_i^T V_j = 0$, i != j
- Listing all eigen vectors in matrix form V
  - $V^T = V^{-1}$
  - $V^T V = I$
  - $V V^T = I$
- $M V_i = \lambda V_i$
- In matrix form : $M V = V \Lambda$
  - Λ is a diagonal matrix with all eigen values
- $M = V \Lambda V^T$

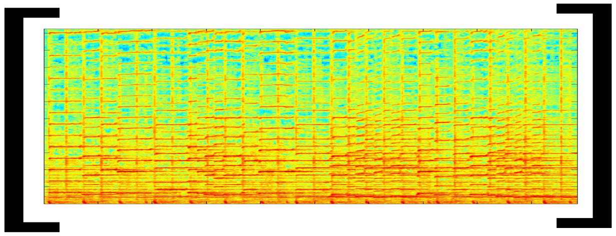
Definiteness..

- SVD: Singular values are always positive!
- Eigen Analysis: Eigen values can be real or imaginary
  – Real, positive Eigen values represent stretching of the space along the Eigen vector
  – Real, *negative* Eigen values represent stretching and *reflection* (across origin) of Eigen vector
  – Complex Eigen values occur in conjugate pairs
- A square (symmetric) matrix is **positive definite** if all Eigen values are real and positive, and are greater than 0
  – Transformation can be explained as **stretching** along orthogonal axes
    - Transformation has no permutation or rotation
  – If any Eigen value is **zero**, the matrix is positive *semi-definite*

Positive Definiteness..

- Property of a positive definite matrix: Defines inner product norms
  - x^TAx is always positive for any vector x if A is positive definite
- Positive definiteness is a test for validity of *Gram* matrices
  - Such as correlation and covariance matrices
  - We will encounter these and other gram matrices later

SVD on data-container matrices



$$\mathbf{X} = [X_1 \; X_2 \quad \cdots X_N]$$

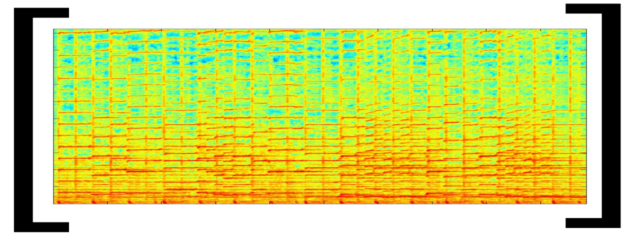
$$\mathbf{X} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}}$$

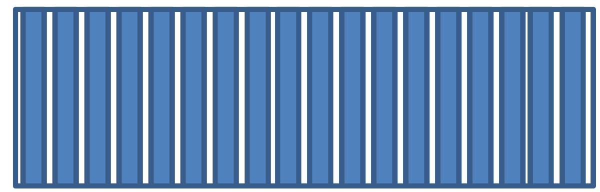
- We can also perform SVD on matrices that are *data containers*
- **S** is a d x N rectangular matrix
  - N vectors of dimension d
- **U** is an orthogonal matrix of d vectors of size d
  - All vectors are length 1
- **V** is an orthogonal matrix of N vectors of size N
- **S** is a d x N diagonal matrix with non-zero entries only on diagonal

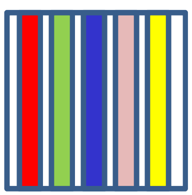
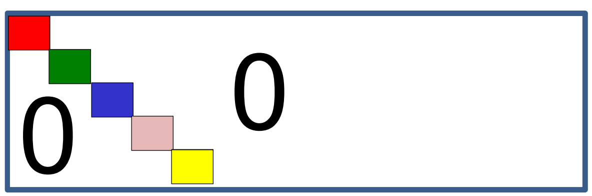
SVD on data-container matrices

$\mathbf{X} = [X_1 \; X_2 \; \cdots X_N]$

$\mathbf{X} = \mathbf{USV}^\mathrm{T}$



$\mathbf{X} =$ 

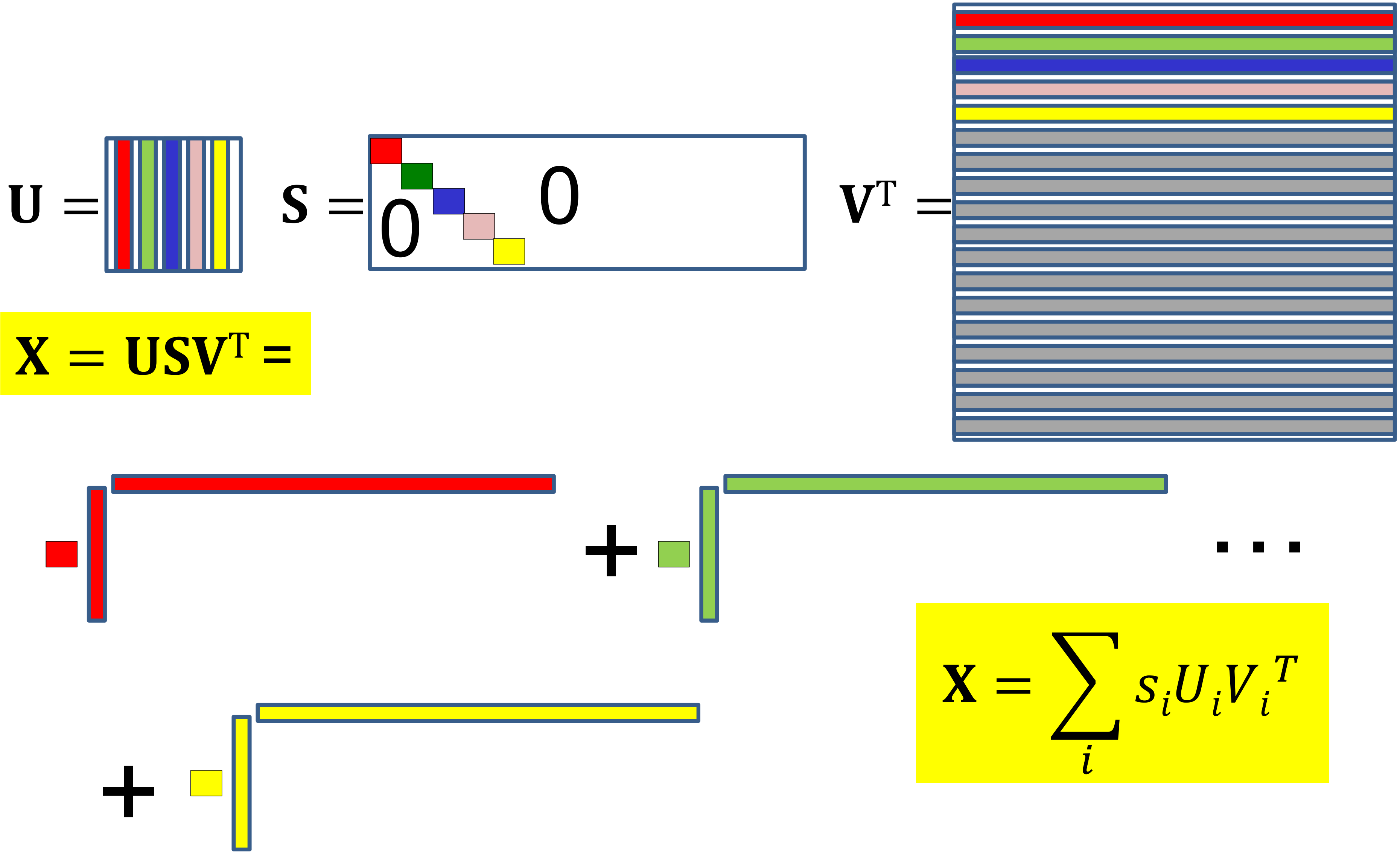
$\mathbf{U} =$  $\mathbf{S} =$ 

$\mathbf{V}^\mathrm{T} =$ 

$|U_\mathrm{i}| = 1.0$ for every vector in $\mathbf{U}$

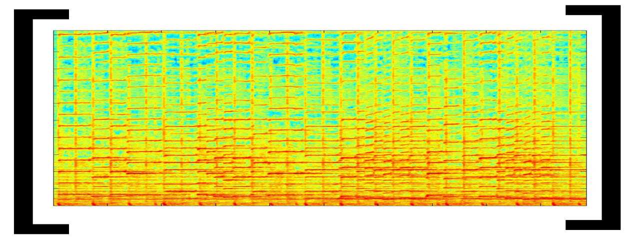
$|V_\mathrm{i}| = 1.0$ for every vector in $\mathbf{V}$

SVD on data-container matrices



$$\mathbf{X} = \sum_i s_i U_i V_i^T$$

Expanding the SVD



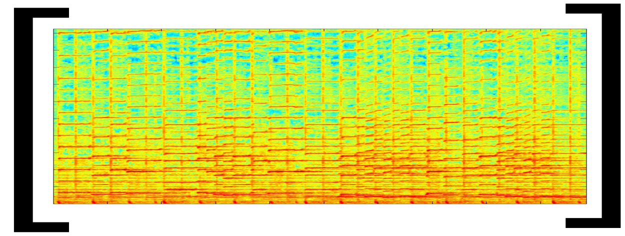
$$\mathbf{X} = [X_1 \; X_2 \quad \cdots X_N]$$

$$\mathbf{X} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}}$$

$$\mathbf{X} = s_1 U_1 V_1^T + s_2 U_2 V_2^T + s_3 U_3 V_3^T + s_4 U_4 V_4^T + \ldots$$

- Each left singular vector and the corresponding right singular vector contribute on “basic” component to the data
- The “magnitude” of its contribution is the corresponding singular value

Expanding the SVD



$$\mathbf{X} = [X_1 \; X_2 \; \cdots X_N]$$

$$\mathbf{X} = \mathbf{USV}^{\mathrm{T}}$$

$$\mathbf{X} = s_1 U_1 V_1^T + s_2 U_2 V_2^T + s_3 U_3 V_3^T + s_4 U_4 V_4^T + \ldots$$

magnitude (s_1) basis (U_1) modulation (V_1^T)

- Each left singular vector and the corresponding right singular vector contribute on "basic" component to the data

- The "magnitude" of its contribution is the corresponding singular value

Expanding the SVD

$$\mathbf{X} = s_1 U_1 V_1^T + s_2 U_2 V_2^T + s_3 U_3 V_3^T + s_4 U_4 V_4^T + \ldots$$

- Each left singular vector and the corresponding right singular vector contribute on "basic" component to the data
- The "magnitude" of its contribution is the corresponding singular value
- Low singular-value components contribute little, if anything
  - Carry little information
  - Are often just "noise" in the data

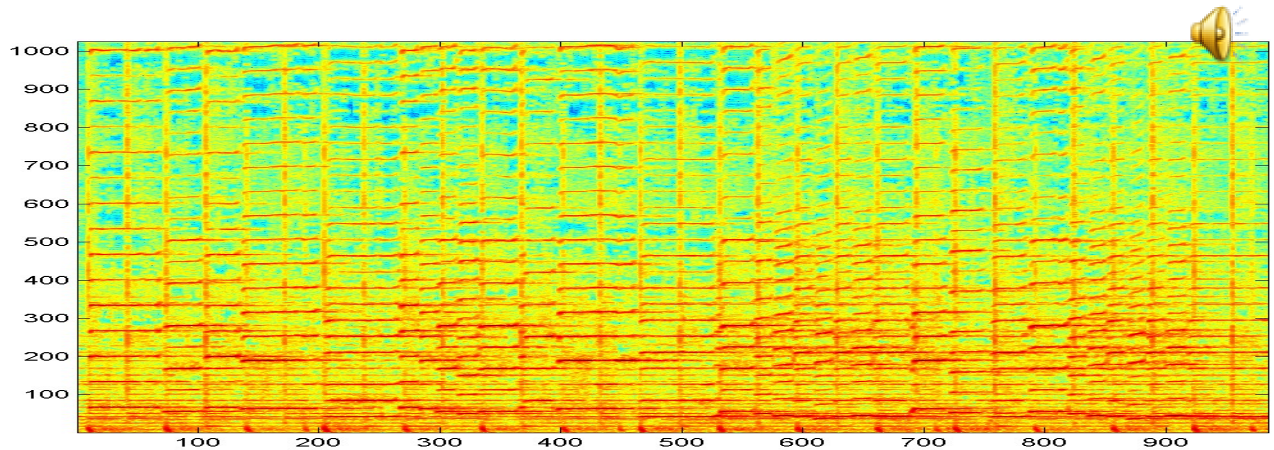
Expanding the SVD

$$\mathbf{X} = s_1 U_1 V_1^T + s_2 U_2 V_2^T + s_3 U_3 V_3^T + s_4 U_4 V_4^T + \ldots$$

$$\mathbf{X} \approx s_1 U_1 V_1^T + s_2 U_2 V_2^T$$

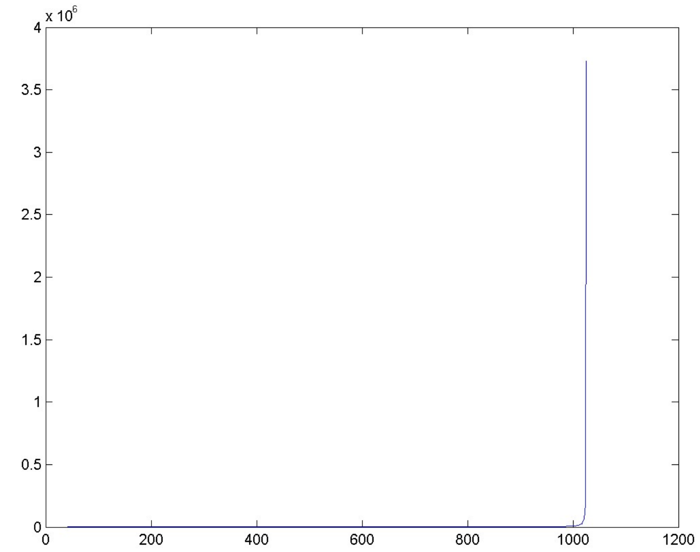
- Low singular-value components contribute little, if anything
  - Carry little information
  - Are often just “noise” in the data
- Data can be recomposed using only the “major” components with minimal change of value
  - Minimum squared error between original data and recomposed data
  - Sometimes eliminating the low-singular-value components will, in fact “clean” the data

An audio example



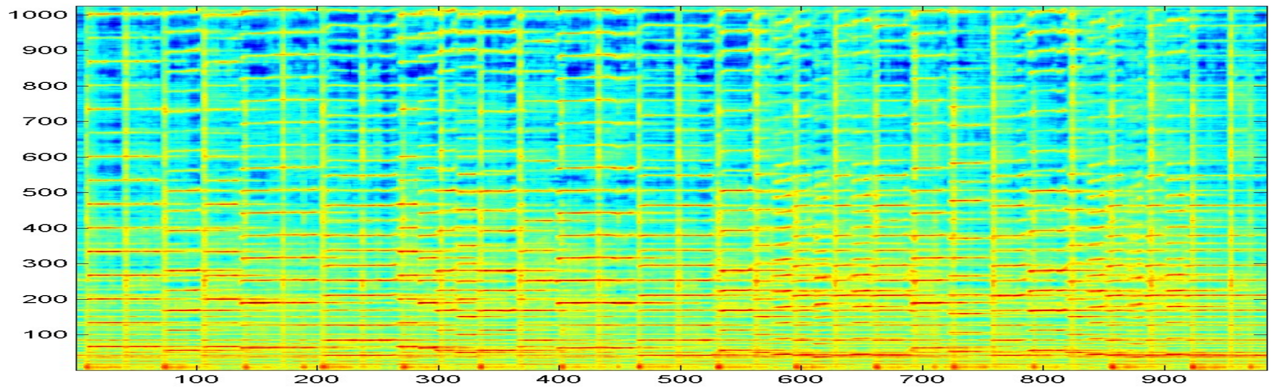
- The spectrogram has 974 vectors of dimension 1025
  - A 1024x974 matrix!
- Decompose: $\mathbf{M} = \mathbf{USV}^T = \sum_i s_i U_i V_i^T$
- **U** is 1024 x 1024
- **V** is 974 x 974
- There are 974 non-zero singular values S_i

Singular Values



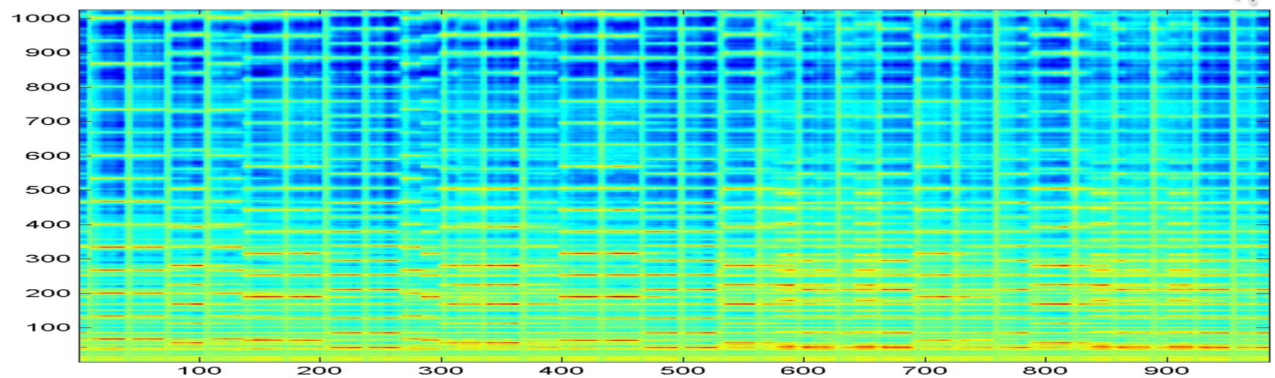
- Singular values for spectrogram **M**
  - Most Singluar values are close to zero
  - The corresponding components are "unimportant"

An audio example



- The same spectrogram constructed from only the 25 highest singular-value components
  - Looks similar
    - With 100 components, it would be indistinguishable from the original
  - Sounds pretty close
  - Background “cleaned up”

With only 5 components



- The same spectrogram constructed from only the 5 highest-valued components
  - Corresponding to the 5 largest singular values
  - Highly recognizable
  - Suggests that there are actually only 5 significant unique note combinations in the music

- Next up: A brief trip through optimization..